PYTHAGORAS AND HIS TRIANGLES

Mita Darbari

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By:

Mita Darbari

First Impression: July 2023

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ISBN: 978-93-93810-76-2

Rs. 1000/- ($80)

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Dedicated to My Husband

Dr. Subodh Darbari, M.S., M.Ch. (Neuro.)

Without whom nothing could have been achieved

PREFACE

Remember Pythagoras?

--New York Times, June 1993

"At Last, Shouts of 'Eureka!' in Age-Old Math Mystery" ran the headlines of a front-page story, courtesy Dr. Andrew Wiles and Fermat's last theorem. When we think of Pythagoras, what immediately comes to our mind is the famous Pythagoras theorem of right-angled triangle that has applications in many branches of modern science: from geometry to Einstein's theory of General Relativity. The impact of this theorem was, or rather still is, so great that it outshines all the other contributions of this great teacher.

The first chapter deals with biography of Pythagoras and some of his teachings. In the second chapter, special Pythagorean Triangles are found after solving quadratic Diophantine equations.

In the third chapter, special Pythagorean Triangles are obtained in relation with cubic and quartic Diophantine equations. The fourth chapter explores the possibility of obtaining the special Pythagorean Triangles in terms of quintic and hexic Diophantine equations.

The fifth chapter deals with heptic and octic Diophantine equations while chapter six obtains special Pythagorean Triangles from nonic and decic Diophantine equations. In chapters 2-6, various properties of these triangles are studied and 3D graphs of these triangles are also plotted.

ACKNOWLEDGEMENT

Any big task can be started alone, but sometimes you cannot make it on your own. It needs the blessings of Almighty in the form of assistance from a chain of persons in the completion of that task. Therefore, it is never gratuitous to thank them profoundly. First and foremost, I owe my deepest gratitude to my supervisors, Late Prof. J.P. Shukla, Former Vice-Chancellor, Rani Durgavati University, Jabalpur and Late Dr. S.S. Rana, Ex-Professor, Department of Mathematics and Computer Science, RDVV, Jabalpur, who have been very kind to give their valuable guidance. I am also highly indebted to them for minutely going through the manuscript and making the necessary changes. Next, this is a great opportunity to express my respect and gratitude to Late Professor T. Pati, Former Vice-Chancellor, University of Allahabad and Shree Jagannath Sanskrit University, Puri, for encouraging me to write on the topic of Pythagoras and going through few of my papers. The present book, Pythagoras and His Triangles, is a part of my thesis.

With fond memories I remember and thank my mother, Late Smt. Karuna Shivastava, for giving me the strength to plod on despite my constitution wanting to give up. I also thank my mother-in-law, Late Smt. Sarla Darbari, who used to be extremely proud of any achievement of mine, however insignificant it may have been!

These acknowledgements would never be complete if I don't mention the invaluable support, understanding, encouragement, concern and help of my husband Dr. Subodh Darbari, in the progress and completion of my research work, who stood like a formidable wall between me and my quitting. And last, but not the least, I am extremely grateful to my loving sons, Prashans and Bhavya for their kind understanding and patience.

Mita Darbari

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ISBN: 978-93-93810-76-2

CHAPTER ONE

PYTHAGORAS - A BRIEF LIFE SKETCH & SOME OF HIS TEACHINGS

Pythagoras in Raphael's *School of Athens*

Janson H. W.; *History of Art*; Fifth Edition; NY; Abrams; 1995; p. 497

1.1 INTRODUCTION

Whenever we think of Pythagoras; what comes to our mind is the famous Pythagoras Theorem of right-angled triangle which still remains the single most important theorem in the history of mathematics and has well over four hundred proofs and their number is still growing [57]. Kepler said,

Geometry has two great treasures: one is the *Theorem of Pythagoras*; the other, the division of a line into extreme and *mean ratio*. The first we may compare to a measure of gold; the second we may name a precious jewel.

This theorem, perhaps, outshines all the other contribution of this great teacher and founder of Pythagorean philosophy based on numbers. As Iamblichus remarks about the life of Pythagoras whom he called as prince and father of this philosophy as a leader,

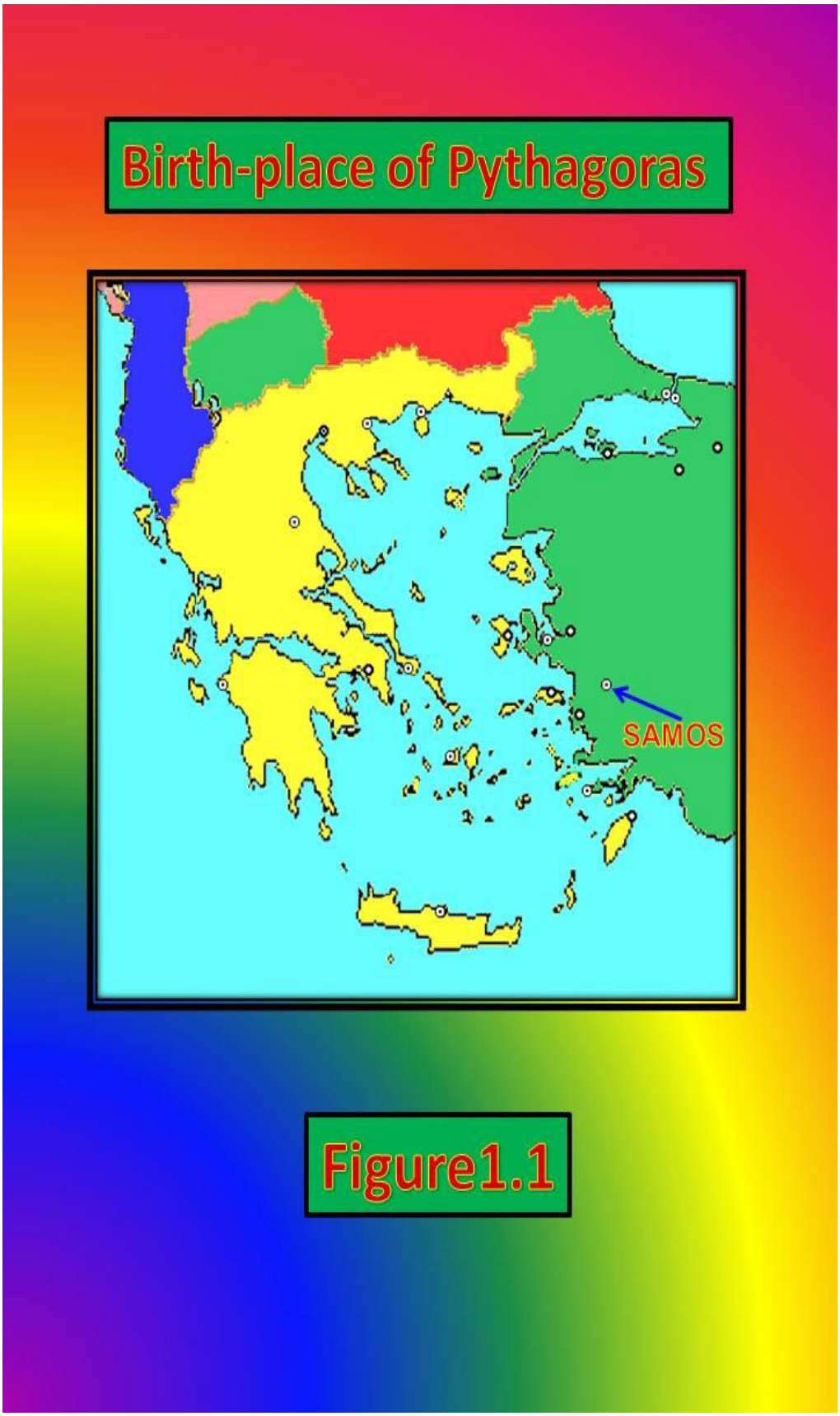
Besides, its beauty and majesty so surpass human capacity, that it cannot be comprehended in one glance. Gradually only can some details of it be mastered when; under divine guidance we approach the subject with a quiet mind [49; Ch. I].

1.2 BIRTH OF PYTHAGORAS

Pythagoras, the son of Pythais and Mnesarchus, who was a seal engraver, was born on the island of Samos, off the coast of Asia Minor (what is now mostly Turkey), about 580 BC. Pythais was a native of Samos. Mnesarchus was a merchant from Tyre. It is said that Mnesarchus was told by a prophet at the temple of Delphi, where he had gone on a business tour, that his wife would present him a son who would surpass all who had ever lived in beauty and wisdom and that he would be of the greatest benefit to the human race in everything pertaining to human achievements [49; Ch. II].

Little is known of Pythagoras's childhood. Pythagoras had, probably, three brothers, Eunestus, Tyrrhenus and Astrasus, who was adopted [62; Ch.10]. As a child Pythagoras spent his early years in Samos but travelled widely with his father. There are accounts of Mnesarchus returning to Tyre with Pythagoras and that he was taught there by the Chaldaeans and the learned men of Syria. It seems that he also visited Italy with his father [62; Ch.1 &2].

ISBN: 978-93-93810-76-2



1.3 HIS TRAVEL & EDUCATION

His father took care that his son should enjoy the best possible education available at that time, studying under Creophilus, then under Phorecydos the Syrian and then under almost all who presided over sacred concerns, to whom he especially recommended his son, including a lute-player, a wrestler and a painter, that he might be as expert as possible in divinity [49; Ch. II & 60; Ch. 11]. Pherekydes (Phorecydos) was one of the most important among his early teachers. The other two, Thales and his pupil, Anaximander, of Miletus introduced him to mathematical ideas- Geometry and astronomy.

He successively associated with each of those philosophers in a manner such that they all loved him, admired his natural endowments and admitted him to the best of their doctrines. Pythagoras was eighteen years old when he first met Thales who was well advanced in age. After communicating to him the utmost he was able to impart to Pythagoras, Thales, laying stress on his advanced age and the infirmities of his body, advised him to go to Egypt to get in touch with the priests of Memphis and Jupiter. Thales confessed that the instruction of these priests was the source of his own reputation for wisdom. Pythagoras was benefited by the instruction of Thales in many respects, but his greatest lesson had been to learn the value of saving time, which led him to abstain entirely from wine and animal food, avoiding greediness, confining himself to nutriments of easy preparation and digestion. As a result, his sleep was short; his soul pure and vigilant and the general health of his body was invariable [49; Ch. II].

With these advantages, first he sailed to Sidon, in Phoenicia where he conversed with the prophets who were the descendants of Moschus, the physiologist and with many others, as well as with the local hierophants. From a desire of and love for contemplation and from an anxiety to miss nothing of the mysteries of the divinities which deserved to be learned he was initiated into all the mysteries of Byblus and Tyre and in the sacred functions performed in many parts of Syria. After gaining all he could from the Phoenician mysteries, following the advice of his teacher Thales, he sailed for Egypt with some Egyptian sailors in the hope of finding monuments of education still more genuine, beautiful and divine. It is said that during the whole trip Pythagoras sat silent where he would be least likely to inconvenience them at their tasks. Moreover, the sailors considered that, contrary to their expectations, their voyage had proceeded without interruptions, as if some deity had been on board [49; Ch. III].

Admission in an Egyptian temple was almost impossible for a foreigner and Pythagoras requested the Polycrates, the ruler of Samos to write to Amasis, the King of Egypt, his friend and former host, to procure him initiation. Coming to Amasis, he was given letters to the priests; of Heliopolis, who sent him on to those of Memphis, on the pretense that they were the more ancient. On the same pretense, he was sent on from Memphis to Diospolis. From fear of the King the latter priests dared not make excuses; but thinking that he would desist from his purpose as a result of great difficulties, enjoined on him very hard precepts, entirely different from the institutions of the Greeks. These he performed so readily that he won their admiration and they permitted him to sacrifice to the Gods and to acquaint himself with all their sciences, a favor theretofore never granted to a foreigner [62; Ch.7 & 8].

Having most solicitously familiarized himself with every detail, he did not, nevertheless, neglect any contemporary celebrity, whether sage renowned for wisdom, or peculiarly performed mystery; he did not fail to visit any place where he thought he might discover something worthwhile. That is how he visited all of the Egyptian priests, acquiring all the wisdom each possessed. He thus passed twenty-two years in the sanctuaries of temples, studying astronomy and geometry and being initiated in no casual or superficial manner in all the mysteries of the Gods. At length, however, he was taken captive by the soldiers of Cambyses and carried off to Babylon. Here he was overjoyed to associate with the Magi, who instructed him in their venerable knowledge and in the most perfect worship of the Gods. Through their assistance,

likewise, he studied and completed arithmetic, music and all the other sciences. After twelve years, about the fifty-sixth year of his age, he returned to Samos [49; Ch. IV]. Pythagoras made a journey to Crete and Sparta shortly after his return to Samos to study the system of laws there.

1.4 PYTHAGOREAN SCHOOL

On his arrival in Samos, he first established a school, which was called, the Semicircle of Pythagoras, in which the Samians consulted about public affairs, feeling the fitness of dispensing justice and promoting profit in the place constructed by him who promoted the welfare of all mankind. Outside of the city he formed a cave adapted to the practices of his philosophy, in which he spent the greater part of day and night, ever busied with scientific research and meditating. His philosophy by then had gained great importance and his fame spread to all Greece so that the best students visited Samos on his account to share in his erudition. But his fellow-citizens insisted on employing him in all their embassies and compelled him to take part in the administration of public affairs. Pythagoras began to realise the impossibility of complying with the claims of his country while remaining at home to advance his philosophy; and observing that all earlier philosophers had passed their life in foreign countries, he determined to resign all political occupations. Pythagoras left Samos and went to southern Italy in about 518 BC (some say much earlier). Pythagoras founded a philosophical and religious school in Croton (now Crotone, on the east of the heel of southern Italy) that had many followers (about six hundred the very first day!) [62; Ch. 20].

Pythagoras was the head of the society with an inner circle of followers known as mathematikoi. The mathematikoi lived permanently with the Society, had no personal possessions and were vegetarians. They were taught by Pythagoras himself and obeyed strict rules. The outer circle of the Society was known as the akousmatics and they lived in their own houses, only coming to the Society during the day. They were allowed their own possessions and were not required to be vegetarians. Both men and women were permitted to become members of the Society, in fact several later women Pythagoreans became famous philosophers. Pythagoras was said to have taught his school 39 years and to have lived a century.

1.5 HIS BELIEFS & TEACHINGS

Of Pythagoras' actual work nothing is known. His school practiced secrecy and communalism making it hard to distinguish between the work of Pythagoras and that of his followers. Pythagoras saw a deep connection between the natural world and numbers and mathematical relations. All is Number was the mystical conclusion of Pythagoreans as he found numbers in geometry, astronomy, music and philosophy. He was the first to use the term philosophy and called himself as Philosophos- the lover of wisdom. He was also the first to use the term cosmos [2, Ch. 14] to indicate that there is order in the universe after his discovery of musical scales [49; Ch. XXVI].

The beliefs that Pythagoras held were that-

(1) At its deepest level, reality is mathematical in nature,

(2) Philosophy can be used for spiritual purification,

(3) The soul can rise to be in union with the divine,

(4) Certain symbols have a mystical significance and

(5) All brothers of the order should observe strict loyalty and secrecy [64]

Besides all this, Pythagoras unfolded to the Greeks all the disciplines, theories and researches that would purify the intellect from the blindness introduced by studies of a different kind, so as to enable it to perceive the true principles and causes of the universe. In addition, the best polity, popular concord, community of possessions among friends, worship of the Gods, piety to the

dead, legislation, erudition, silence, abstinence from eating the flesh of animals, continence, temperance, sagacity, divinity and in one word, whatever is anxiously desired by the scholarly, was brought to light by Pythagoras [49; Ch.VI].

Pythagoras believed in the transmigration of soul and hence in the kinship of all living things. He taught that the soul is immortal, doomed to a cycle of rebirth but could be liberated by means of ritual purification.

Thus, by education and good fortune he became the most beautiful and godlike of all those who have been celebrated in the annals of history.

1.6 PYTHAGOREAN NUMBER THEORY (NUMBER ABSOLUTE- ARITHMETIC)

Counting is the first step of mathematics and Number Theory is undoubtedly one of the oldest branches of mathematics, perhaps as old as civilization. Since the dawn of history numbers and their properties have fascinated humanity. Ancient Babylonian clay tablets have been found containing integer solutions to Pythagoras's equation $a^2 + b^2 = c^2$, systematically listed with all the arithmetic carried out in base 60[5].

By tradition, most of the first two books of Euclid's Element can be attributed to the Pythagoreans. If this is the case, then much of what is still taught to adolescents around the world today as geometry, number, space and algebra is essentially Pythagorean. The Pythagorean philosophy rested on the assumption that whole number is the cause of the various qualities of man and matter. This led to an exaltation and study of number properties and arithmetic (considered as the theory of numbers), along with geometry, music and spherics (astronomy) was studied in the Pythagorean School. Pythagoras and his followers, in conjunction with the fraternity's philosophy, took the first steps in the development of number theory and at the same time laid much of the basis of future number mysticism. The Pythagorean School distinguished between odd and even numbers, created the concept of divisibility and the concept of perfect and amicable numbers and knew about the series {1, 3, 6, 10...} of triangular numbers, created by adding succeeding integers to their predecessors.

1.7 CLASSIFICATION OF NUMBERS

Odd and Even Numbers

Numbers were displayed in the form of gnomons, which were "L" shaped carpenter's squares. The odd series, as we can see plainly in FIGURE 1.2, has regularity to it while the even series does not have so (FIGURE 1.3).

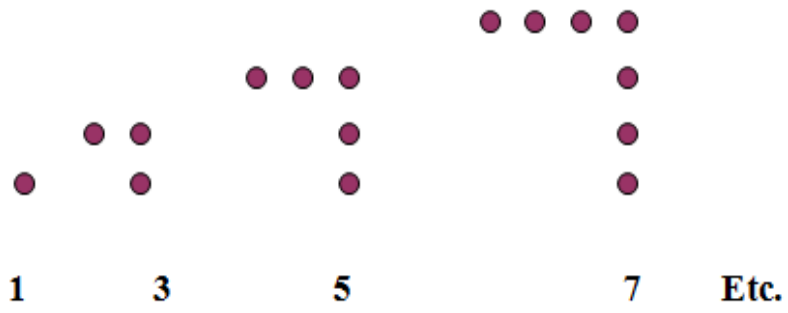


Figure 1.2- Odd Numbers

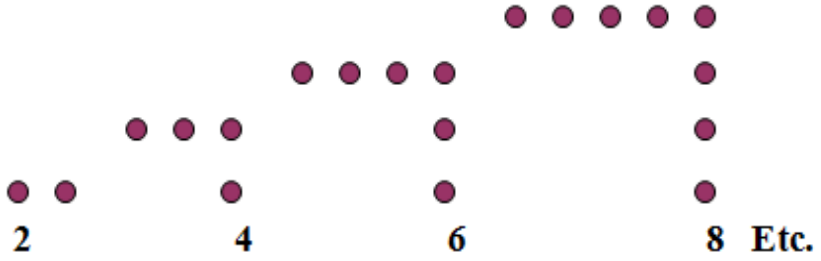


Figure 1.3: Even Numbers

Perfect Numbers

Pythagoras and his followers studied perfect numbers, more for their mystical properties than for their theoretic properties. There were "perfect" numbers of two kinds. The first kind included only 10, which was basic to the decimal system and the sum of the first four numbers 1 + 2 + 3 + 4 = 10. The second kind of "perfect" numbers were those equal to the sum of their proper divisors. A number is deficient if it exceeds the sum of its proper divisors and abundant if it is less than the sum of its proper divisors. Deficient number is also called imperfect. E.g., 8, whose factors (1, 2 and 4) add up to 7, is imperfect.

Between one and 10,000 there are only four perfect numbers- 6, 28, 496 and 8,128. Although perfect numbers are regarded as arithmetical curiosities, their study has helped to develop the theory of numbers. Euclid proved that a number of the form $(2^n-1)\times 2^{n-1}$ is a perfect number if the factor 2^n-1 is prime, n being a natural number.

Prime Numbers

It will be millions of years before we'll have any understanding and even then, it won't be a complete understanding, because we're up against the in finite. - P. Erdos [13].

Pythagoras also developed the concept of the prime number, a number that cannot be evenly divided by any other number.

Prime numbers are considered the "building blocks" of the natural numbers because every single natural number, excluding the number 1, is either a prime number or a product of prime numbers. In other words, every number is either a prime or can be factored into a product of primes.

Amicable Numbers

Amicable numbers are a pair of numbers with the following property: the sum of all of the proper divisors of the first number (not including itself) exactly equals the second number while the sum of all of the proper divisors of the second number (not including itself) likewise equals the first number. The set of 220 and 284 was the first known set of amicable numbers. Iambilic of Chalcis ascribed the discovery of these numbers to Pythagoras [72]:

284 = 1x284 = 2x142 = 4x71

220 = 1x220 = 2x110 = 4x55 = 5x44 = 10x22 = 11x20

Adding the factors as though they are perfect numbers (all but the number itself), we find that 1 + 2 + 4 + 71 + 142 = 220 and

1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284.

Pythagoras discovered the relationship and coined the term amicable because he considered the numbers to be a symbol of friendship. No other pairs were known in the West until 1636 when Fermat discovered 17,296 and 18,416 as a second pair. Today, there are over 5000 known pairs. Interested Mathematicians are researching on Amicable Triples and quadruples.

1.8 FIGURATE NUMBERS

Figurate number can be defined as a number which can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a regular polygon the number is called a polygonal number. These numbers, considered as the number of dots in certain geometrical configurations, represent a link between geometry and arithmetic. Some examples are:

Triangular Numbers

The numbers, which can be arranged in a compact triangular pattern, are termed as triangular numbers (Figure 1.4). These are formed by partial sum of the series $T_n = 1+2+3+4+5+6+7...+n$. The n^{th} triangular number can be obtained as $T_n = n \times (n+1)/2$, where n is any natural number.

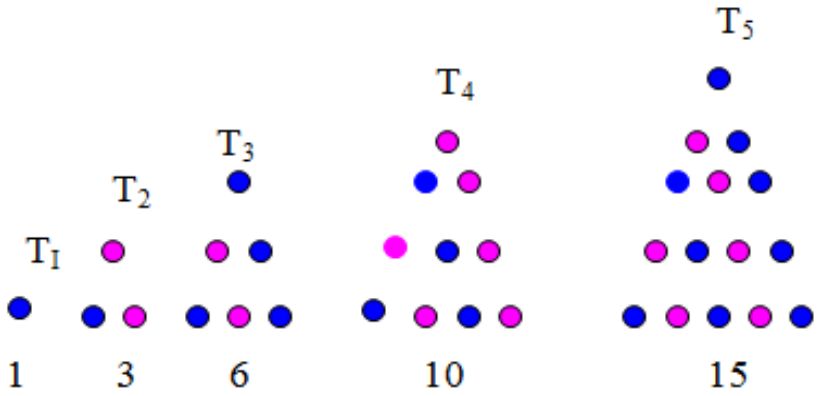


Figure 1.4

So, $T_1 = 1$

$T_2 = 1 + 2 = 3$

$T_3 = 1 + 2 + 3 = 6$

$T_4 = 1 + 2 + 3 + 4 = 10$

$T_5 = 1 + 2 + 3 + 4 + 5 = 15$

Flocks of birds often fly in this triangular formation. Even several airplanes when flying together constitute this formation. Ancient Greek mathematicians first studied the properties of such numbers, particularly the Pythagoreans and their remarkable properties continue to fascinate mathematicians and they continue to discover more properties [36]. Triangular numbers most commonly arise in probabilistic situations, most notably in establishing the number of outcomes for given events.

Square and Oblong numbers

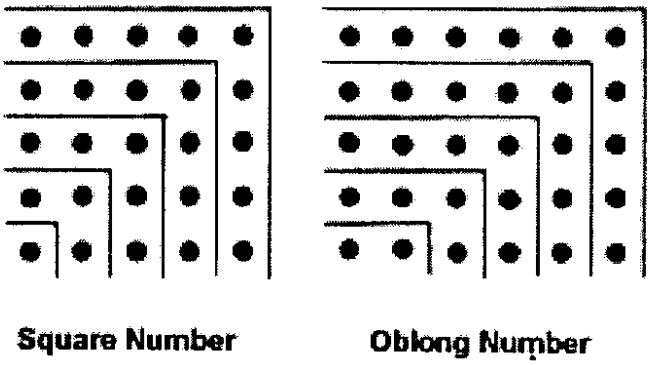


Figure 1.5

An Oblong number can be arranged in a rectangle whose width and height differ by one unit; for simplicity let them be wider than tall. So, whereas the Square numbers have sizes 1 X 1, 2 X 2, 3 X 3,

4 X 4, etc., the Oblong numbers have sizes 1 X 2, 2 X 3, 3 X 4, etc. (FIGURE 1.5). Square numbers are the simplest example of a quadratic function. Quadratic functions are used frequently to model real world situations where the rate of growth or decline is not constant. Pythagoreans noticed that twice triangular numbers are the Oblongs, 2 X 1 = 2, 2 X 3 = 6, 2 X 6 = 12, etc. [5]. One can also see this geometrically; the Oblong number 12 can be divided into twice the triangular 6 (Figure 1.6).

Figure 1.6

Pentagonal Numbers

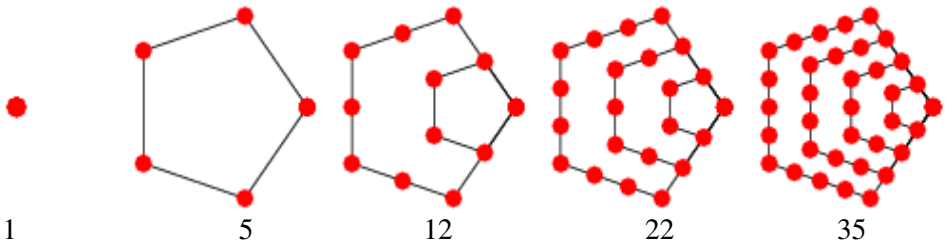
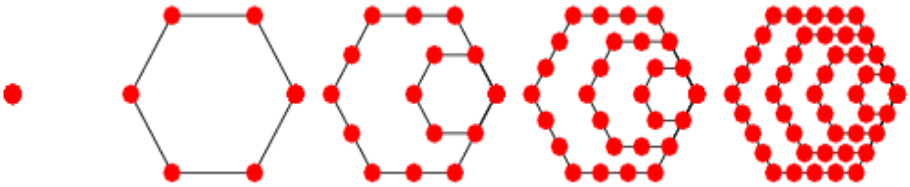


Figure 1.7

A polygonal number of the type n (3n-1) /2 is a pentagonal number (Figure 1.7). The first few are 1, 5, 12, 22, 35, 51, 70 ... (Sloane'sA000326) [71]**.** Every pentagonal number is 1/3 of a triangular number.

Hexagonal Numbers



1 6 15 28 45

Figure 1.8

A polygonal number of the type n (2n −1) is a hexagonal number (Figure 1.8). The first few are 1, 6, 15, 28, 45, 66, 91, (Sloane'sA000384) [71]. Every hexagonal number is a triangular number since n (2n-1) = (1/2) (2n-1) [(2n-1) + 1].

In 1830, Legendre proved that every number larger than 1791 is a sum of four hexagonal numbers, and Duke and Schulze-Pillot [12] improved this to

three hexagonal numbers for every sufficiently large integer. Fermat proposed that every number is expressible as at most k k-gonal numbers (Fermat's polygonal number theorem). Cauchy proved the proposition completely in 1813.

1.9 PYTHAGOREAN TRIPLES & TRIANGLES

The Pythagoreans discovered that the sum of certain pairs of square numbers is also a square number. The numbers whose squares add up to another square—such as 3, 4 and 5 or 5, 12 and 13—are called Pythagorean triples. There are infinitely many Pythagorean triples and there is an elegant geometric construction using the unit circle that gives a general formula for all of them [11].

These triples have a geometric interpretation in the form of famous Pythagoras Theorem; usually expressed as $\mathbf{a^2 + b^2 = c^2}$. If a, b and c are natural numbers and satisfy Pythagorean equation, then (a, b, c) is called a Pythagorean Triangle. A solution of Pythagorean Equation is called Primitive if a, b and c are natural numbers and have no common divisor greater than 1.

1.10 PYTHAGOREAN QUADRUPLES AND QUINTUPLES

After finding all solutions to the equation $a^2 + b^2 = c^2$, it is natural to try to extend the result. A Diophantine Equation is an equation in one or more variables for which integer solutions are sought.

A Pythagorean quadruple is a set of positive integers **a, b, c and d** which satisfy the following Diophantine equation – $\mathbf{a^2 + b^2 + c^2 = d^2}$. Example – (2, 6, 9, 11) i.e., $\mathbf{2^2 + 3^2 + 6^2 = 7^2}$ or 4 + 9 +36 = 49.

A Pythagorean quintuple is a set of positive integers **a, b, c, d** and **e** which satisfy the following Diophantine equation – $\mathbf{a^2 + b^2 + c^2 + d^2 = e^2}$. Example - (3, 4, 12, 84, 85) i.e., $\mathbf{3^2 + 4^2 + 12^2 + 84^2 = 85^2}$.

A four-dimensional version of the Pythagorean Theorem can be seen in the Separation Formula in Relativity, i.e., $\mathbf{t^2 - (x^2 + y^2 + z^2)/c^2 = s^2}$. This equation has integer solutions if the velocity of light (**c**) is itself set to an integer value, i.e., **1**, which would mean that units of time are years and units of distance are light-years. If $\mathbf{x^2 + y^2 + z^2}$ are replaced with its equivalent, $\mathbf{r^2}$ then the equation reduces to $\mathbf{t^2 - r^2 = s^2}$ which is simply have a version of the Pythagorean Theorem itself. Otherwise, we will need a Pythagorean "quintuplet." If **c = 1**, we can give a solution of $\mathbf{t^2 - (x^2 + y^2 + z^2) = s^2}$ as $\mathbf{85^2 - (3^2 + 4^2 + 12^2) = 84^2}$. Thus, with Pythagorean "quadruplets" and "quintuplets", we have integer solutions to the three and four-dimensional versions of the Pythagorean Theorem.

Further extending the equation of Pythagorean triplets, $a^3 + b^3 + c^3 = d^3$ and $a^4 + b^4 + c^4 + d^4 = e^4$ it was found that there exist several solutions to these equations [4]. There are parametric formulae, which give infinite sets of solutions, but not necessarily all the solutions [39]. For the 2nd equation, the smallest solution is $30^4 + 120^4 + 272^4 + 315^4 = 353^4$.

1.11 FERMAT'S LAST THEOREM

Pierre de Fermat (1601-1665) started pondering about Pythagoras' equation about right triangles i.e.; $a^2 + b^2 = c^2$, while he was studying Bachet's translation of Diophantus's 'Arithmetica', an ancient work on number puzzles. As Fermat contemplated on this equation, he began hypothesizing on a related one, $a^3 + b^3 = c^3$. Through trial and error, he soon found that finding integers that made this equation true seemed to be impossible. After trying powers of 4 and 5 and being equally frustrated, he wrote the following notes in the wide margin of that ancient text besides a description of Pythagoras's Theorem:

… it is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as a sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvelous demonstration of this proposition, which this margin is too narrow to contain [76].

For more than 300 years mathematicians have attempted to find the proof or explanation that Fermat didn't write- Fermat's Last Theorem. But the effort of these mathematicians has not been in vain as they definitely helped in the development of mathematics.

The only case which we know that Fermat proved is n = 4. Euler proved for the case n = 3, using arithmetic in the ring theory. Subsequent work by Dirichlet, Legendre, Kummer and many others settled many more exponents, at the same time creating most of modern algebraic number theory and algebra. By 1987, the Theorem was known to be true for all $n \geq 150000$. In 1986, an unexpected connection was found, by Frey, between the Fermat equation and another class of Diophantine equation called Elliptic curves. A solution to Fermat's equation would lead to the existence of an elliptic curve with properties so strange that they would contradict widely believed, but then unproved, conjectures about elliptic curves. Ribet proved this connection. Finally, the American mathematician Andrew Wiles with the help of Richard Taylor proved the elliptic curve conjecture and thereby firmly establishing the truth of Fermat's Last theorem, though it took him eight years. The proof was published in Annals of Mathematics in 1995, 330 years after Fermat's death, but obviously it was not Fermat's proof, since Wiles used mathematics, which was not available in the 17th century.

1.12 INCOMMENSURABLE

It was fundamental to the beliefs of the early Pythagoreans that all things (abstract, natural, human affairs) were understandable by considering the intrinsic properties of Natural numbers and their ratios. This was the foundation of the faith and wisdom of Pythagoras and essential to this faith was the concept of commensurability. Heath quotes the first scholium on Book X of the Elements [41] as stating:

".. the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it by means of their observation of numbers...They called all magnitudes measurable by the same measure commensurable"

For example, the number 8 can be measured in twos - the oblong number 4 x 2, 12 can be also measured in twos - the oblong number 6 x 2, so 8 and 12 are commensurable as they both can be measured in twos. Similarly with geometry, they reasoned that given any two line segments of unequal length, it should always be possible to find a third line segment, perhaps very small, that can be marked off a whole number of times into each of the given segments. With the concept of commensurability, the early Pythagoreans could confidently base their faith on "all is

number". If any pair of numbers were found to be incommensurable then the philosophical foundation of the Order would be threatened. It is generally accepted that such a threat did eventuate in Pythagoras' time and that it arose in the investigation of the cherished Pythagorean Theorem.

The discovery, which created such a threat to Pythagoreanism, was that natural numbers or their ratio is not sufficient when comparing the length of the diagonal of a square to its side. It is not possible to find a Natural number, which measures both the diagonal and the side of a square. Similarly, it is not possible to find a small length which can be marked off a whole number of times into both the diagonal and the side. The belief that all the secrets of the Universe would be found in Natural numbers was in tatters. There existed, contrary to intuition, incommensurable numbers and line segments. We call the numbers needed to represent the length of the diagonal, the irrational numbers meaning - "unable to be expressed as a ratio". The Pythagorean proof is still as elegant and breathtaking in its logical structure today as it was then. The proof "**reductio ad absurdum**" is also called "Proof **by Contradiction**".

Perhaps he reasoned as follows. Suppose that the square root of two can be expressed as a/b where this fraction is in lowest terms. That is, a and b, have no common factor. (1, 1, a/b) form the sides of a right triangle. Multiply each of the sides by b to get (b, b, a), which is also a right triangle and is in fact a Pythagorean triple. Because the two short sides are the same, they must both be even or odd. This obvious fallacy means that we are then forced to conclude that the square root of two cannot be expressed as a/b, it is irrational. The Pythagoreans gave a name to these new numbers-the unutterable, or the alogos, which had a double meaning- not a ratio and not to be spoken. The Pythagoreans therefore denied the existence of irrational number that they themselves discovered [6].

Thus, early Pythagoreans have been credited with the discovery of the irrational number, though it is not clear whether they had expanded their investigations into any other surds. The discovery of incommensurables introduced method of indirect proof and, apparently in this connection, by the development of the definitional system of mathematics. In fact, the proof of irrationality promoted a stricter approach to geometry for it showed that the evident and trustworthy do not necessarily coincide, as it could not be accepted that some lengths were incommensurable with rational numbers. And so, it was decided that numbers could not be associated with lengths. This decision led to a division between arithmetic and geometry that was not reconciled until the time of Descartes.

A very important consequence of the discovery was that it made the real number system complete.

1.13 THE PYTHAGOREAN THEORY OF PROPORTION

Pythagoras is also given the credit of discovering the theory of proportion. In Babylon Pythagoras had probably learned the three basic means, the arithmetic, the geometric and the subcontrary which was called the harmonic later on.

Let a > b > c and b be the mean of a and c, then the three means are:

(a − b) ÷ (b − c) = a ÷ a,	arithmetic	**a + c = 2b,**
(a − b) ÷ (b − c) = a ÷ b,	geometric	**$ac = b^2$,**
(a − b) ÷ (b − c) = a ÷ c,	subcontrary	**1/a + 1/c = 2/b.**

In fact, Pythagoras or more probably the Pythagoreans added seven more proportions.

Denoting arithmetic, geometric and harmonic by A, G and H respectively, the Pythagoreans called the proportion, **A: G= G: H** the perfect proportion. The proportion

a: A = H : c or **a : (a + c)/2 : 2ac/(a + c) : c,** was called the musical proportion.

1.14 PYTHAGOREAN ASTRONOMY

According to Pythagoras, the position of each body in the universe was determined by the essential dignity of that body. The popular concept of his day was that the earth occupied the center of the solar system; that the planets, including the sun and moon, moved around the earth; and that the earth itself was flat and square. Contrary to this concept and regardless of criticism, Pythagoras first proposed a non-geocentric system, perhaps partly on the basis of moral and religious grounds: to them, humanity and earth were imperfect. He affirmed that fire was the most important of all the elements; that the center was the most important part of every body; and that, just as Vesta's fire was in the midst of every home, so in the midst of the universe was a flaming sphere of celestial radiance. He called this central globe the Tower of Jupiter, the Globe of Unity, the Grand Monad and the Altar of Vesta, poetically called the "Hearth of the Universe" or "Throne of Zeus". The sun is a glass sphere which catches and reflects this hearth-light [38].

The universe is spherical in shape and finite in size. Outside it is infinite void which enables the universe to breath. As the sacred number 10 symbolized the sum of all parts and the completeness of all things, it was only natural for Pythagoras to divide the universe into ten spheres, symbolized by ten concentric circles. These circles began at the center with the globe of Divine Fire. In this central fire is located the governing principles, the force which directs the movement and the activities of the universe. In the universe there revolve in circles the following bodies. Nearest to the central fire revolves the counter-earth, another mysterious planet, called Antichthon, which always accompanies the earth, the orbit of the earth coming next to that of the counter-earth; next to the earth, reckoning in order from the centre outwards comes the moon, next to the moon the sun, next to the sun the five plants and last of all, outside the orbits of the five plants, the sphere of the fixed stars. The counter-earth, which revolves in a small orbit, is not seen by us because the hemisphere of the earth on which we live is turned away from counter-earth. He recognised that the orbit of the Moon was inclined to the equator of the Earth and he was one of the first to realise that Venus as an evening star was the same planet as Venus as a morning star. He recognised that the earth was a sphere, probably more because he believed that a sphere was the most perfect shape than for genuine scientific reasons. For the same reason Pythagoras would surely hold that the sun, the moon and the other heavenly bodies are also spherical in shape.

It is probable that he was the first to state the view (attributed to Alcameon and some of the mathematicians) that the planets as well as the sun and moon have a motion of their own from west to east opposite to and independent of the daily rotation of the spheres of the fixed stars from east to west. [40]

1.15 PYTHAGORAS DISCOVERY OF MUSICAL SCALE

The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic.

- Gottfried Wilhelm von Leibnitz [55]

Music and Mathematics have an unusual relation. Music is the only art form, where the form and the medium are the same. Mathematics is the only science where the methods and the subject are the same. Mathematics is the study of mathematics by means of mathematics. Music is only created and experienced as music. Thus, there is a natural connection between mathematics and music: Both are experienced as pure objects of the brain and both have meaning outside of the brain only by artificial connections.

Pythagoras is credited for the discovery of musical scales. One day while meditating upon the problem of harmony, Pythagoras chanced to pass a brazier's shop where workmen were pounding out a piece of metal upon an anvil**,** producing sounds that harmonized, except one. But he recognized in these sounds, the concord of the octave, the fifth and the fourth. He saw that the sound between the fourth and the fifth, taken by itself, was a dissonance and yet completed the greater sound among them. Delighted, therefore, to find that the thing he was anxious to discover had by divine assistance succeeded, by noting the variances in pitch between the sounds made by large hammers and those made by smaller implements and carefully estimating the harmonies and discords resulting from combinations of these sounds, he gained his first clue to the musical intervals of the diatonic scale [54; XXVI].

Four integers are said to be in musical progression when they are in the ratio a: 2ab/ (a + b): (a + b)/2: b, for example, 6, 8, 9 and 12 are in musical progression [66]. The key to harmonic ratios is hidden in the famous Pythagorean tetractys made up of the first four numbers- 1, 2, 3 and 4, which in their proportions reveal the intervals of the octave, the diapente and the diatessaron [38].

Pythagoras went a few steps further however and he acknowledged that twelve (12) was the "most divisible" small number and that these basic ratios can be expressed in regards to the number twelve [69].

12:12 (unison), 12:6 (octave), 12:8 (fifth), 12:9 (fourth).

He therefore concluded that twelve (12) was the most ideal musical number; thousands of years of musical composition can attest to the notion that he was mostly right. 12 is also double of the first perfect number 6, the third triangular number.

This is the way in which music is said to have been discovered by Pythagoras. Having reduced it to a system, he delivered it to his disciples to utilize it to produce things as beautiful as possible.

1.16 MUSIC OF SPHERES- RELATION BETWEEN MATHEMATICS, MUSIC AND ASTRONOMY

Having once established music as an exact science, Pythagoras applied his newly found law of harmonic intervals to all the phenomena of Nature, even going so far as to demonstrate the harmonic relationship of the planets, constellations and elements to each other. Pythagoras conceived of three kinds of music: 1. Musica Instrumentalis or music made by people playing instruments or singing. 2. Musica Humana or the music which human beings are, e.g., he likened people's bodies to the body of a guitar and our nervous systems to the strings. 3. Musica Mundana or the Music of the Spheres [1].

Pythagoras believed that the planets each produce a musical tone and their orbits were laid out in proportion like the notes in a musical scale. Together they create a Celestial Harmony. The Pythagoreans believed most people couldn’t hear this harmony of the spheres because they had grown accustomed to it from birth, but that Pythagoras alone could indeed hear it.

[62; Ch.30] In the Pythagorean concept of the music of the spheres, the interval between the earth and the sphere of the fixed stars was considered to be a diapason--the most perfect harmonic interval. The allowing arrangement is most generally accepted for the musical intervals of the planets between the earth and the sphere of the fixed stars: From the sphere of the earth to the sphere of the moon; one tone; from the sphere of the moon to that of Mercury, one half-tone; from Mercury to Venus, one-half; from Venus to the sun, one and one-half tones; from the sun to Mars, one tone; from Mars to Jupiter, one-half tone; from Jupiter to Saturn, one-

half tone; from Saturn to the fixed stars, one-half tone (Figure 1.10). The sum of these intervals equals the six whole tones of the octave [38].

And the trouble with people is that the music which they are (Musica Humana) gets out of harmony with the Music of the Spheres. This is the root cause of disease, anger, depression, anxiety -in short, all human ills. Therefore, he composed musical remedies, tunes which affected the human organism (Musica Humana) in such a way as to bring it back into harmony with the Music of the Spheres.

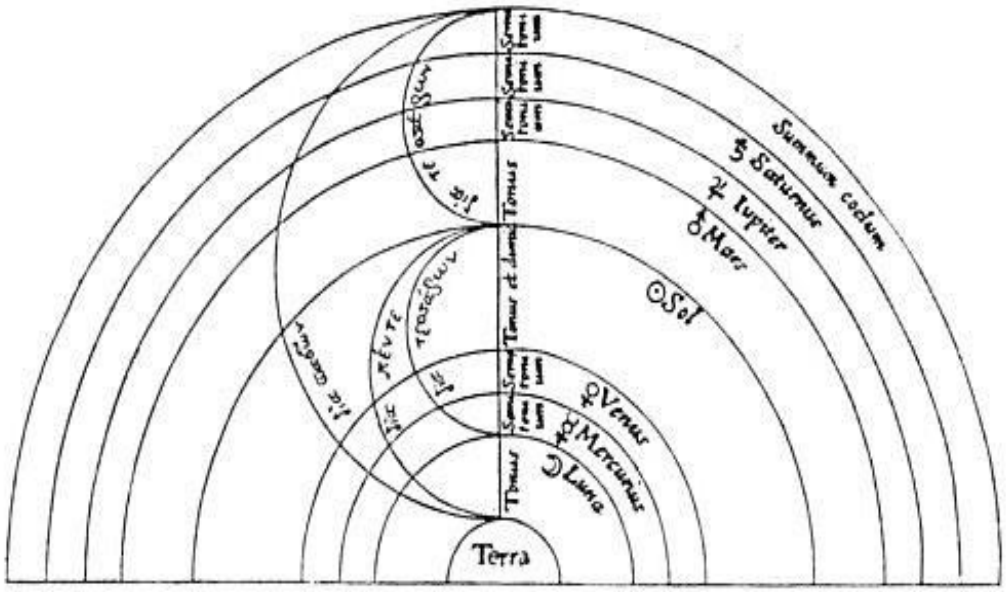


Figure 1.10: The Intervals and Harmonies

1.17 MUSIC AS PURIFICATION

Pythagoras was likewise of opinion that music, if properly used, greatly contributed to health and used it in no careless way, but as purification. Indeed, he restricted this word to signify music used as medicine. About the vernal season he used a melody in this manner. In the middle was placed a person who played on the lyre and seated around him in a circle were those able to sing. Then the lyrist in the centre struck up and the singers raised certain paeans, through which they were evidently so overjoyed that their manners became elegant and orderly. This music instead of medicines was also used at certain other times [49; Ch.25].

Certain melodies were devised as remedies against the passions of the soul, as also against despondency and gnashing of the teeth, which were invented by Pythagoras as specifics. Further, he employed other melodies against anger and rage and all other aberrations of the soul. Another kind of modulation was invented against desires. He likewise used dancing, which was accompanied by the lyre, instead of the pipe, which he conceived to have an influence towards insolence, being theatrical and by no means liberal. For the purpose of correcting the soul, he also used selected verses of Homer and Hesiod. Thus, through music Pythagoras produced the most beneficial correction of manners and lives.

1.18 PYTHAGOREAN TETRAKTYS

The mysterious Pythagorean tetraktys which was composed of ten dots arranged in four rows to form a triangle (FIGURE 1.11) was the sacred symbol upon which the Pythagoreans took their most binding oath:

I swear by him who the Tetraktys found, Whence all our wisdom springs and which contains Perennial Nature's fountain, cause and root [74].

 ISBN: 978-93-93810-76-2

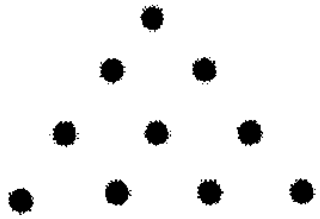


Figure 1.11

Theon of Smyrna says that the Pythagoreans honored this symbol "because it appears to contain the nature of all things." According to Iamblichus, the Pythagorean Tetraktys had eleven forms, each one applying to some one particular phase of cosmic or terrestrial life.

The first quaternary is 1, 2, 3, 4 as 1 + 2 + 3 + 4 = 10. The second is unity, the side, the square and the cube. The third is the point, the line, the surface and the solid. The fourth is fire, air, water and earth. The fifth is the pyramid, the octahedron, the icosahedron and the cube. The sixth is the seed, the length, the width and the height. The seventh is man, the family, the village and the city. The eighth is thought, science, opinion and sense. The ninth is the rational, the emotional, the willful parts of the soul and the body. The tenth is spring, summer, autumn and winter. The eleventh is childhood, adolescence, maturity and old age. And the perfect world which results from these quaternaries is geometrically, harmonically and arithmetically arranged, containing in power the entire nature of number, every magnitude and everybody, whether simple or composite. It is perfect because everything is part of it and it is itself a part of nothing else. This is why the Pythagoreans used the oath and through which all things are assimilated to number [43].

1.19 TRANSMIGRATION OF SOUL, MUSIC AND MATHEMATICS

The main purpose of the Pythagorean Order was the cultivation of holiness, entailing high thinking and low living at all times as we have seen that the core of their religious doctrine was that the soul, the pure and noble part of us, is mired in the body, from which it longs to escape and as a consequence, Pythagoreans engaged in various practices that were intended to purify the soul so that it could escape from the body. He also preached that men are already divine; we achieve immortality not by becoming divine, but by realizing that which is already within us. Pythagoras believed that philosophy was the path to the discovery of this divine nature, the purified soul.

But his philosophy was All is Number and how was it related to transmigration of soul? The answer might have been the music! We have seen in Section 1.16 that he was Pythagoras who discovered that musical harmonics are governed by mathematics. Pythagoras might have applied this discovery into his theory of transmigration of souls. The soul of the music is the mathematical structure of its harmony which is eternal and the body, through which it is expressed, is the musical instrument which is perishable. Just the mathematical idea of music can be expressed through many instruments; the soul can be reincarnated in many bodies and just as the soul is immortal, the idea of music exists eternally although the instrument, through which it is expressed, may decay.

1.20 CONCLUSION

For the first time in the history the study of mathematics rose above the needs of commerce and was studied for its own sake by Pythagoras and his followers. They systemized the study of numbers for its future development. The discovery of irrational numbers by the Pythagoreans was a big achievement for mathematics and not something the unutterable or the alogos as they

called them. The irrational numbers helped in making the real number system complete without which the study of calculus and analysis would have been impossible. Pythagoreans also gave to the world a strong method of proof - "**reductio ad absurdum**", also called "Proof **by Contradiction**". The techniques that were developed to solve a number of Diophantine Equations including Fermat's Last Theorem have led to the development of much of modern algebra and number theory. This is typical of progress in Number Theory, where there have been many cases of entirely new mathematical theories being created to solve special cases and often quite elementary-seeming problems. Number theory, which was once notorious for having no real-world applications, is being used, in recent decades, as both prime factorization (in the RSA algorithm) and Diophantine problems (in algorithms using elliptic curves) have become essential components of public-key encryption systems. Thus, we see that, Pythagoras and his followers, who studied the elementary properties of natural numbers first, paved a way to the development of Number Theory and laid a strong foundation for pure Mathematics. In fact, the ancient Pythagorean philosophy that "all is number" is literally true in our modern digitized world.

Pythagoras's discovery of the arithmetical basis of the musical intervals was not just the beginning of musical theory; it was the beginning of science. For the first time, man discovered that universal truths could be explained through systematic investigation and the use of symbols such as mathematics. The Pythagoreans believed that everything which existed had a voice and that all creatures were eternally singing the praise of the Creator. Basically, each living being on every scale is oscillating, for example, celestial bodies, nature and its numerous entities, humans, molecules and atoms - probably also the nucleus. If one can imagine that notion, then each atom has its signature sound according to its number of electrons. The atoms sing, as other bodies do on another and bigger scale. Together all sing their different songs in union - the Universe. Thus therefore, through music Pythagoras produced the most beneficial correction of manners and lives. Once that window was opened, the light spread across the whole breadth of human curiosity, not least in the field of cosmogony and ultimately in quantum gravity. "The genius of Pythagoras lay in the comprehensive way he joined the inner man and the cosmos" [50].

Pythagoras taught that the soul was immortal and that after death it transmigrated into other animated bodies. After certain specified periods, the same events occur again; that nothing was entirely new. And therefore, all animated beings were kin and should be considered as belonging to one great family. Clearly, whether it turns out to be what really happens to us after death or not, a belief in reincarnation makes for a kinder, gentler, more compassionate world. Pythagoras must be admired for trying to make it so.

ISBN: 978-93-93810-76-2

CHAPTER TWO

SPECIAL PYTHAGOREAN TRIANGLES IN RELATION WITH QUADRATIC DIOPHANTINE EQUATIONS

CHAPTER 2 (A)

SPECIAL PYTHAGOREAN TRIANGLES WITH PERIMETER AS A SUM OF THREE SQUARES*

2.1 INTRODUCTION

In the Pythagorean Mathematics, Gopalan and Janaki [19] have given perimeter of Pythagorean Triangles as a pentagonal Number. Integral solutions of ternary quadratic equation are given by Gopalan, Somnath & Vanitha [28]. Special Pythagorean Triangles are generated by Gopalan & Vijyalakshmi [32] and Gopalan and Devibala [17]. Pythagorean triangles with their perimeters as the sum of three distinct squares are studied by Gopalan and Devibala [18]. All these studies motivated us to examine whether there is any Pythagorean Triangle with two consecutive sides and sum of its legs a square. In this chapter, we make an attempt to find patterns of Special Pythagorean Triangles with two consecutive sides and their perimeters as sum of three distinct squares.

2.2 METHOD OF ANALYSIS

We recall the definition of Pythagorean Triangle.

Definition 2.2.1: If X, Y and Z are natural numbers and satisfy Pythagorean equation,

$$X^2 + Y^2 = Z^2, \qquad \textbf{(2.2.1)}$$

then (X, Y, Z) is called a Pythagorean Triangle. X and Y are called its two legs and Z is called its hypotenuse.

Definition 2.2.2: A solution of Pythagorean Equation is called Primitive if X, Y and Z are natural numbers and have no common divisor greater than one, i.e., (X, Y, Z) = 1.

For some integers m, n of opposite parity such that m > n > 0 and (m, n) = 1, that is, they have no common divisor, the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$,

is given by Niven & Zuckerman [60] where

$X = m^2 - n^2$,

$Y = 2mn$,

$$Z = m^2 + n^2. \qquad \textbf{(2.2.2)}$$

***Sum of the legs is a square:** We consider the case when the sum of two legs is a perfect square. In that case,

$\exists\ \beta \in \mathbf{N}$ such that

$$X + Y = \beta^2, \qquad \textbf{(2.2.3)}$$

which by equation (2.2.2) becomes

$$m^2 - n^2 + 2mn = \beta^2. \qquad \textbf{(2.2.4)}$$

*This Paper is published in International Journal of Emerging Technologies and Innovative ResearchVol. 6. Issue 2, 135-139, February 2019.

Gopalan & Devibala [18] have generated m, n and β of equation by (2.2.4)

$m = 2p^2 + q^2 - 2pq$, $n = 2pq$, $\beta = 2p^2 - q^2$, where p and q are positive integers. **(2.2.5)**

Equations (2.2.2) and (2.2.5) give

$X = (2p^2 + q^2)(2p^2 + q^2 - 4pq)$,

$Y = 4pq(2p^2 + q^2) - 8p^2 q^2$,

$Z = (2p^2 + q^2 - 2pq)^2 + (2pq)^2$. **(2.2.6)**

***Hypotenuse and one leg are consecutive:** In such case,

either $Z = X + 1$or $Z = Y + 1$

Now, $Z \neq X + 1$, for if $Z = X + 1$, then by equation (2.2.2) we get

$m^2 + n^2 = m^2 - n^2 + 1$

$\Rightarrow 2n^2 = 1$

$\Rightarrow$ n is an irrational number,

which contradicts n being a natural number.

Assuming $Z = Y + 1$, then equation (2.2.2) gives

$m^2 + n^2 = 2mn + 1$

$\Rightarrow (m - n)^2 = 1$

$\Rightarrow m = n + 1$. **(2.2.7)**

Substituting the values of m and n from equation (2.2.5), we get the following equation

$2p^2 + q^2 - 4pq = 1$. **(2.2.8)**

Equation (2.2.8) gives the value of q as

$q = 2p \pm \sqrt{(1 + 2p^2)}$. **(2.2.9)**

From equations (2.2.6) and (2.2.8), we get

$X = 1 + 4pq$,

$Y = 4pq(1 + 2pq)$,

$Z = 4pq(1 + 2pq) + 1$. **(2.2.10)**

***Perimeter is Sum of Three Squares:**
From equations (2.2.2) and (2.2.3), we get perimeter as

$X + Y + Z = \beta^2 + m^2 + n^2$. **(2.2.11)**

Equations (2.2.2), (2.2.5), (2.2.9) and (2.2.10) generate (X, Y, Z) which satisfy equations (2.2.1) and (2.2.3) in correspondence with equation (2.2.11).

Few examples of such Pythagorean triangles, each of whose hypotenuse and a leg are consecutive, i. e.

$Z = Y + 1$, and perimeter is a sum of three distinct squares, i.e.,

$X + Y + Z = m^2 + n^2 + \beta^2$, are given in the Table 2a.1 below while Table 2a.2 gives the verification of these Primitive Pythagorean Triangles.

Table 2a.1: Special Pythagorean Triangles (X, Y, Z)

S.N.	p	q	m	n	X	Y	Z
1	2	1	5	4	9	40	41
2	2	**7**	29	28	57	1624	1625
3	12	**7**	169	168	337	56784	56785
4	12	**41**	985	984	1969	1938480	1938481
5	70	**41**	5741	5740	11481	65906680	65906681
6	70	**239**	33461	33460	66921	2239210120	2239210121
7	408	**239**	195025	195024	390049	76069111200	76069111201
8	408	**1393**	1136689	1136688	2273377	2584121492064	2584121492065
9	2378	**1393**	6625109	6625108	13250217	87784125273544	87784125273545
10	**?**	**8119**	**?**	**?**	**?**	**?**	**?**

Table 2a.2: Verification of $X + Y = \beta^2$ and $X + Y + Z = \beta^2 + m^2 + n^2$

S.N.	$X + Y = \beta^2$	β	$X + Y + Z = \beta^2 + m^2 + n^2$
1	$9 + 40 = 49 = 7^2$	**7**	$90 = 9 + 40 + 41 = 7^2 + 5^2 + 4^2$
2	$57 + 1624 = 41^2$	**41**	$3306 = 57 + 1624 + 1625 = 41^2 + 29^2 + 28^2$
3	$337 + 56784 = 57121 = 239^2$	**239**	$113906 = 337 + 56784 + 56785 = 239^2 + 169^2 + 168^2$
4	$1969 + 1938480 = 1940449 = 1393^2$	**1393**	$3878930 = 1393^2 + 985^2 + 984^2$
5	$11481 + 65906680 = 65918161 = 8119^2$	**8119**	$131824842 = 8119^2 + 5741^2 + 5740^2$
6	$66921 + 2239210120 = 2239277041 = 47321^2$	47321	$4478487162 = 47321^2 + 33461^2 + 33460^2$
7	$390049 + 76069111200 = 76069501249 = 275807^2$	275807	$152138612450 = 275807^2 + 195025^2 + 195024^2$
8	$2273377 + 2584121492064 = 2584123765441 = 1607521^2$	1607521	$5168245257506 = 1607521^2 + 1136689^2 + 1136688^2$
9	$87784138523761 = 13250217 + 87784125273544 = 9369319^2$	9369319	$175568263797306 = 9369319^2 + 6625109^2 + 6625108^2$
10	**?**	**?**	**?**

2.3 OBSERVATIONS

We observe that

1. $X + Y + Z = 2 (1 + 4pq) (1 + 2pq) = 0 (\text{mod } 2)$.
2. The Pythagorean Triangles (X, Y, Z) given by (2.2.8) are all primitives as $Z = Y + 1$.
3. $(Y + Z – X)^2 = 2(Y + Z) (Z – X)$.
4. $(X + 2Y + Z)^2 = (Z – X)^2 + 4(X + Y) (Y + Z)$.
5. $X + 2Y + Z \pm 2\{ (X + Y) (Z - Y)\}^{1/2} = 0 \text{ (mod16) or } = 0 (\text{mod4})$.
6. Except for $q = 1$, the values of β and the values of q are same with q repeated twice.
7. For every value of p, there are two values of q and vice-versa.

Thus, referring to (?) in row 10 of Table 1 and Table 2, the question arises: whether q can be obtained from previously obtained β and hence p?

Putting $q = 8119$ in equation (2.2.8) i.e.,

$2p^2 + q^2 - 4pq = 1$

We get, p = 2378 and p = 13860.

Solving for m, n, X, Y, Z and β when p = 2378 using software **Mathematica,** we get

m = 38613965,

n = 38613964,

X = 77227929,

Y = 2982076508814520,

Z = 2982076508814521 and

β = 54608393

Which satisfy (2.2.11) as shown in Table 2a.3.

Table 2a.3: The Values of (?) in Table 2a.1 and Table 2a.2

S.N.	p	q	m	n	X	Y
10	2378	8119	38613965	38613964	77227929	2982076508814520

S.N.	Z	B	$X + Y = \beta^2$	$X + Y + Z = \beta^2 + m^2 + n^2$
10	2982076508814521	54608393	77227929 + 2982076508814520 = 54608393^2	5964153094856970 = $54608393^2 + 38613965^2 + 38613964^2$

2.4 3D PLOTS

Solving the Diophantine equation $m^2 - n^2 + 2mn = \beta^2$ i.e.,

$X + Y = \beta^2$ to get Pythagorean Triangles (X, Y, Z) where perimeter is a sum of three distinct squares with the help of **Mathematica** for n < m; 0 < m < 10000; 0 < n < 10000;

0 < β < 10000, we get 9354 solutions, out of which only 1259 are primitives.

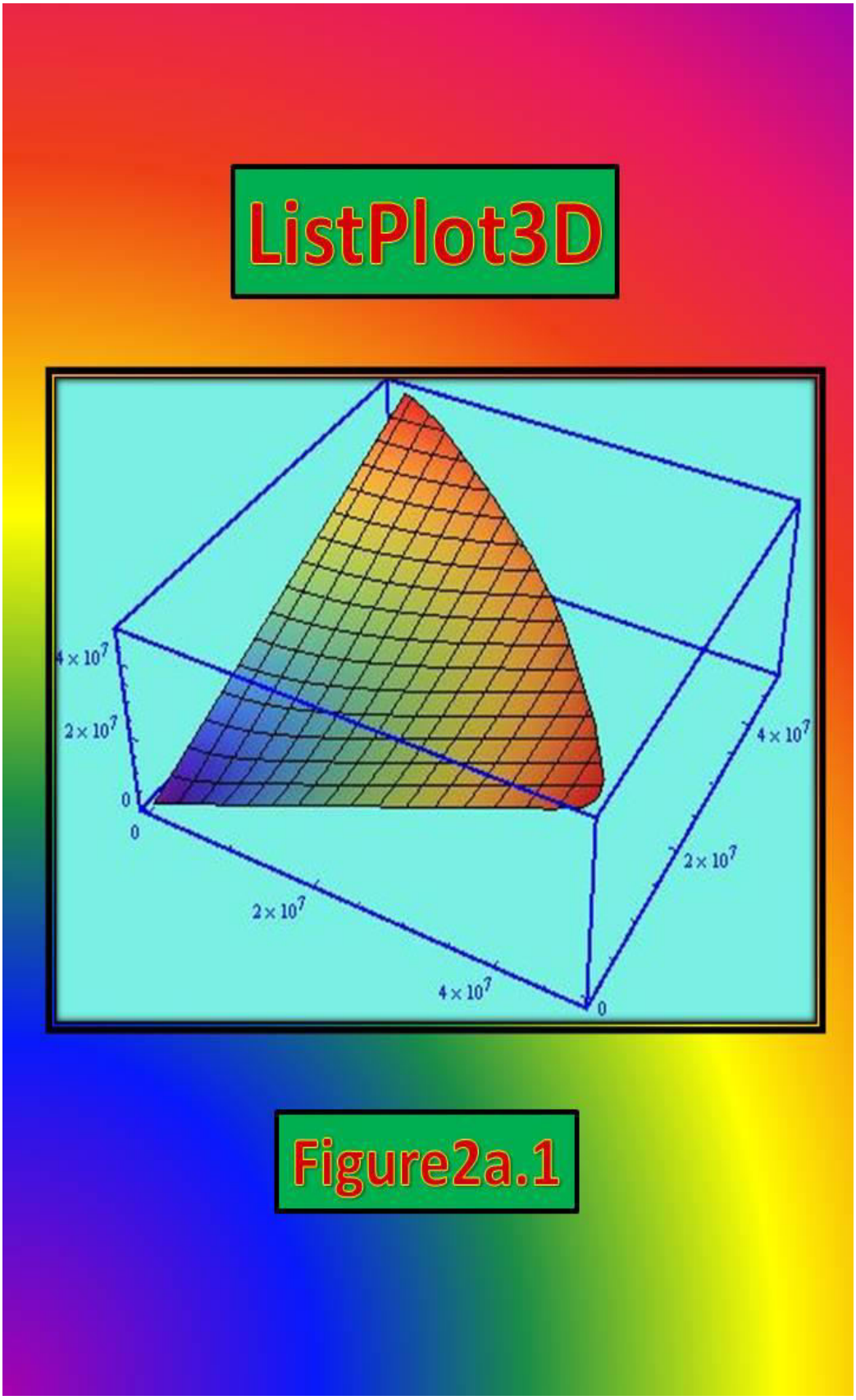
ListPlot3D: ListPlot3D [{(x_1, y_1, z_1), (x_2, y_2, z_2)…}] generates a plot of the surface with heights zi at positions (x_i, y_i).

ListSurfacePlot3D: ListSurfacePlot3D [{(x_1, y_1, z_1), (x_2, y_2, z_2)…}] plots a three-dimensional surface constructed to fit the specified points.

ListPointPlot3D: ListPointPlot3D [{(x_1, y_1, z_1), (x_2, y_2, z_2)…}] generates a 3D scatter plot of points with coordinates (x_i, y_i, z_i).

The ListPlot3D, ListSurfacePlot3D and ListPointPlot3D of 9354 values of (X, Y, Z) and ListPointPlot3D of 1259 primitive values of (X, Y, Z) are given below:

ISBN: 978-93-93810-76-2



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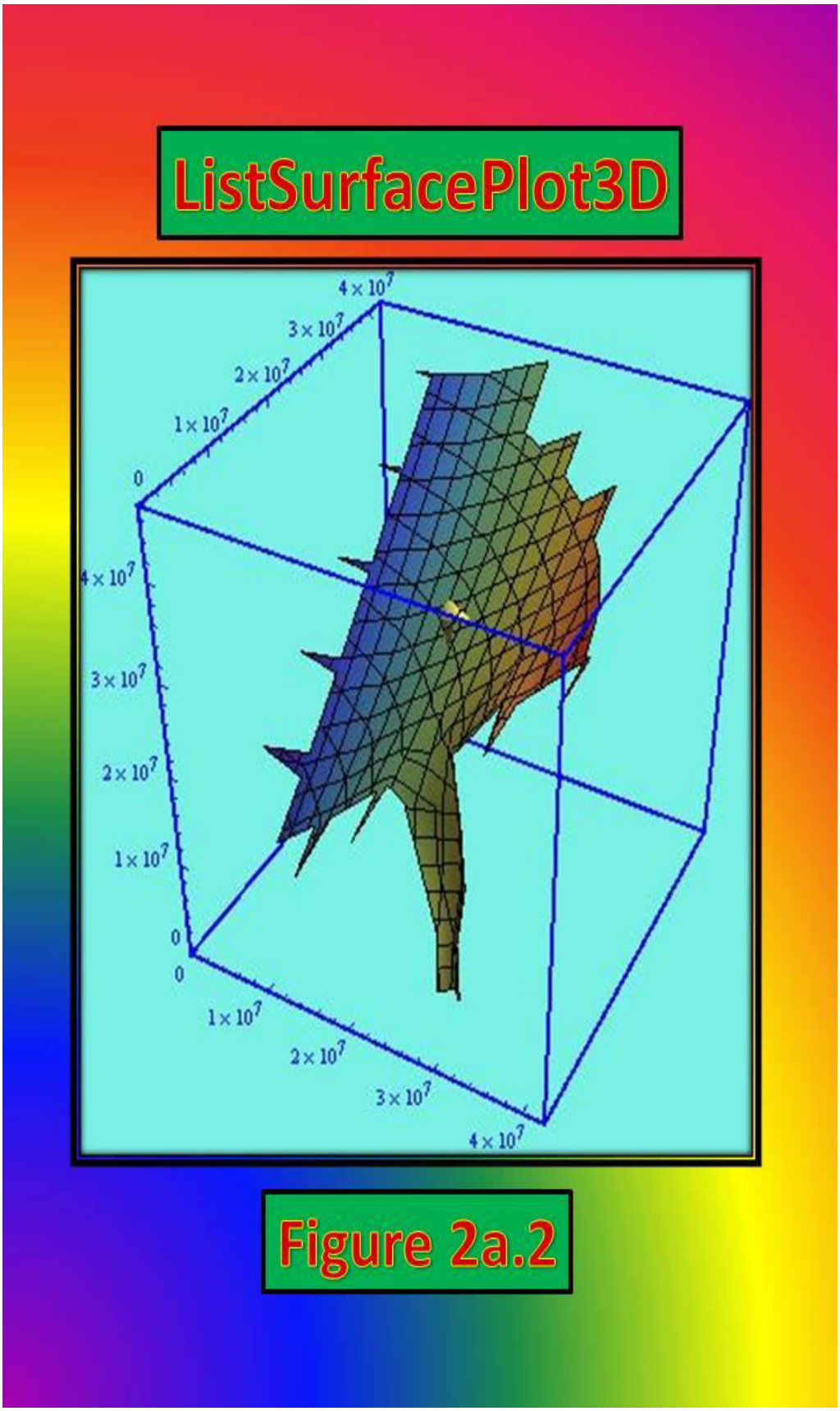

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ListPointPlot3D

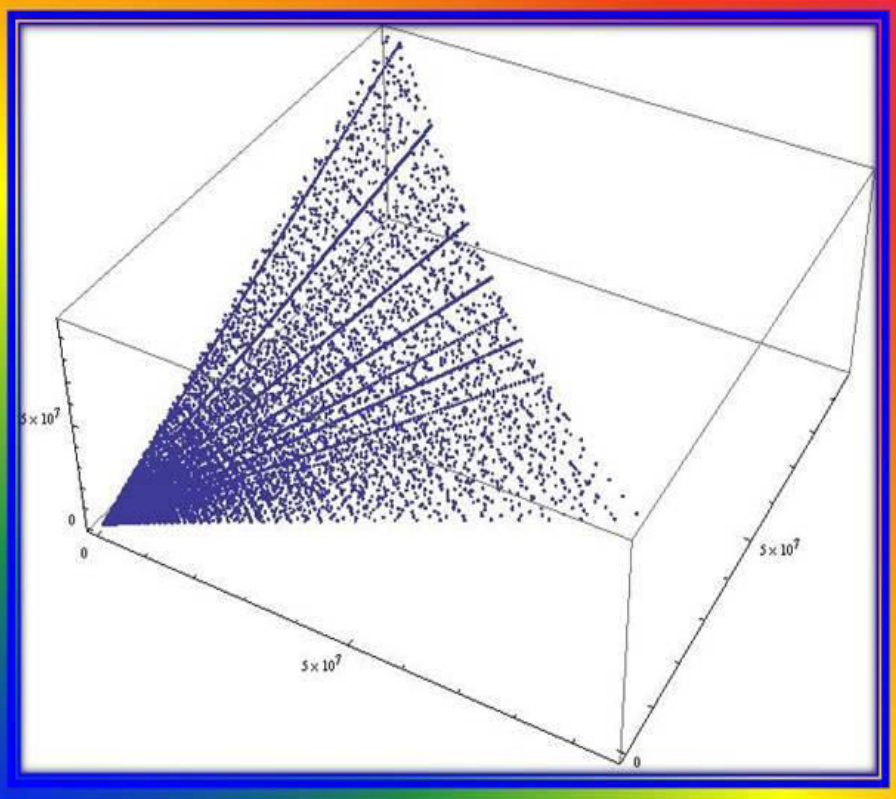


Figure 2a.3

ISBN: 978-93-93810-76-2

ListPointPlot3D
Primitive (X,+ Y, Z)

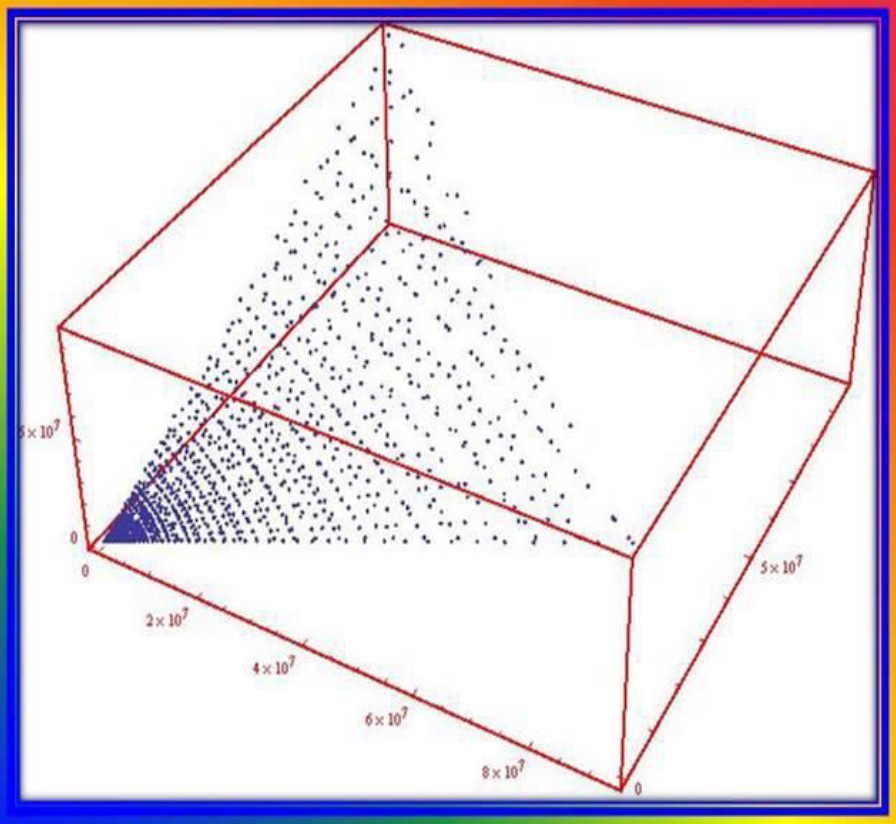


Figure 2a.4

CHAPTER 2(B)

SPECIAL PYTHAGOREAN TRIANGLES IN TERMS OF TRIANGULAR NUMBERS*

2.5 INTRODUCTION

The study of integral solutions of various quadratic Diophantine equations are done by Gopalan, Somnath and Vanitha [30], Janaki and Vidyalakshmi [53], Gopalan and Sivakami [27] and Gopalan and Vijyasankar [33]. Gopalan and Srividhya [31] have studied the relation among M-gonal numbers in terms quadratic equation. Howards [44] states two theorems: **Any square number is the sum of two successive triangular numbers** and **the sum of any number of consecutive odd integers, starting with 1, is a perfect square**, which motivated us to examine whether there exists any Pythagorean Triangle, with two consecutive legs, related to Triangular numbers. In this chapter, an attempt has been made to find patterns of Special Pythagorean Triangles, with two consecutive legs, in terms of Triangular Numbers.

2.6 METHOD OF ANALYSIS:

Definition 2.6.1: A number given by $T_n = n \times (n+1)/2$ is called a Triangular Number, where n is any natural number.

Now, as seen in the previous chapter, the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$, **(2.6.1)**

is given by Niven & Zuckerman [60] where

$X = |m^2 - n^2|$, $Y = 2mn$, $Z = m^2 + n^2$ **(2.6.2)**

for some positive integers m and n of opposite parity such that (m, n) = 1.

***Two legs are consecutive**: In the Primitive solutions of Pythagorean triangle, one of the legs is even and the other is odd. Y = 2mn is obviously even. Then,

$X - Y = \pm 1$

Taking $Y = 2mn$, $X = Y + 1$ **(2.6.3)**

$\Rightarrow 2mn + 1 = m^2 - n^2$

$\Rightarrow m^2 - n^2 - 2mn - 1 = 0$

$\Rightarrow (m - n - 1)(m - n + 1) = 2n^2$

$\Rightarrow (m - n - 1)(m - n + 1)$ is even

$\Rightarrow (m - n - 1)$ and $(m - n + 1)$ are both even.

Let $(m - n - 1) = 2r$, where $r > 0$.

Then, $(m - n + 1) = 2r + 2$.

Now, $(m - n - 1)(m - n + 1) = 2n^2$

$\Rightarrow 2r(r + 1) = n^2$ **(2.6.4)**

$\Rightarrow n^2$ is even

*This Paper has been published in International Journal of Physical Sciences: Ultra Scientist; 23(2) A (2011); 559- 662.

$\Rightarrow$ n is even.

***m and n in terms of Triangular Numbers:**
As n is even, we assume,

$n = 2p.$ **(2.6.5)**

From (2.6.4),

$n^2 = 4p^2 = 2r\ (r + 1)$

Then,

$p^2 = r\ (r + 1) / 2$

$\Rightarrow$ p^2 is a triangular number.

Equations (2.6.2) and (2.6.3) give

$m^2 - 4mp - 4p^2 - 1 = 0$

$\Rightarrow m = 2p \pm \sqrt{(8p^2 + 1)}.$ **(2.6.6)**

Equation (2.6.6) gives the values of m in terms of p where p^2 is a triangular number. Each value of p^2 will give two values of p which will correspond to two values of m and hence two values of X and Y whereas only the positive value of p generates n. Choosing perfect squares from series of Triangular Numbers,

we get the following triangular numbers:

$p^2 = r_1 = 1^2$; $r_8 = 36 = 6^2$; $r_{49} = 1225 = 35^2$; $r_{288} = 41616 = 204^2$;

$r_{1681} = 1189^2$; $r_{9800} = 6930^2$; etc. **(2.6.7)**

Equations (2.6.2), (2.6.5) and (2.6.6) generate X and Y which satisfy (2.6.1) in correspondence with (2.6.7).

Few examples of these Pythagorean triangles (X, Y, Z) are given in the Table 2b.1 below:

Table 2b.1: Special Pythagorean Triangles (X, Y, Z)

p^2	p	m	n	X	Y	Z
1	-1	1	2	3	4	**5**
	1	**5**	2	21	20	**29**
36	-6	**5**	12	119	120	**169**
	6	**29**	12	697	696	**985**
1225	-35	**29**	70	4059	4060	**5741**
	35	**169**	70	23661	23660	33461
41616	-204	**169**	408	137903	137904	195025
	204	**985**	408	803761	803760	1136689
?		**985**				

2.7 OBSERVATIONS

We observe that

1. p = -1 gives all the three sides consecutive.
2. Except for m = 1, the values of Z and the values of m are same with m repeated twice.

Thus, the question arises: whether p can be obtained from previously obtained Z and hence m? From (2.6.6) we get

$(m - 2p)^2 = 8p^2 + 1$

$\Rightarrow 4p^2 + 4mp - m^2 + 1 = 0$

$\Rightarrow p = \{-m \pm \sqrt{(2m^2 - 1)}\} / 2.$

Therefore, for m = 985, $p^2 = 204^2$ & 1189^2 and

for m = 5741, $p^2 = 1189^2$ & 6930^2.

Solving equation (2.6.6) with the help of software Mathematica, we find that. for $0 < m < 10^{20}$, there are just 53 values for p and m! These 53 values for p and m are given in Table 2b.2:-

Table 2b.2: Values of p and m when $0 < m < 10^{20}$

S.N.	p	m
1	-1	1
2	-6	5
3	1	5
4	-35	29
5	6	29
6	-204	169
7	35	169
8	-1189	985
9	204	985
10	-6930	5741
11	1189	5741
12	-40391	33461
13	6930	33461
14	-235416	195025
15	40391	195025
16	-1372105	1136689
17	235416	1136689
18	-7997214	6625109
19	1372105	6625109
20	-46611179	38613965
21	7997214	38613965
22	-271669860	225058681
23	46611179	225058681
24	-1583407981	1311738121
25	271669860	1311738121
26	-9228778026	7645370045
27	1583407981	7645370045
28	-53789260175	44560482149
29	9228778026	44560482149
30	-313506783024	259717522849
31	53789260175	259717522849
32	-1827251437969	1513744654945
33	313506783024	1513744654945
34	-10650001844790	8822750406821
35	1827251437969	8822750406821

36	-62072759630771	51422757785981
37	10650001844790	51422757785981
38	-361786555939836	299713796309065
39	62072759630771	299713796309065
40	-2108646576008245	1746860020068409
41	361786555939836	1746860020068409
42	-12290092900109634	10181446324101389
43	2108646576008245	10181446324101389
44	-71631910824649559	59341817924539925
45	12290092900109634	59341817924539925
46	-417501372047787720	345869461223138161
47	71631910824649559	345869461223138161
48	-2433376321462076761	2015874949414289041
49	417501372047787720	2015874949414289041
50	-14182756556724672846	11749380235262596085
51	2433376321462076761	11749380235262596085
52	-82663163018885960315	68480406462161287469
53	14182756556724672846	68480406462161287469

The above values of p and m show that there are just 53 Pythagorean Triangles, with two consecutive sides, in terms of Triangular numbers when $0 < m < 10^{20}$!

CHAPTER 2(C)

SOME SPECIAL PYTHAGOREAN TRIANGLES*

2.8 INTRODUCTION

Mathematicians all over the ages have been fascinated by the problem of finding all Pythagorean triangles whose perimeters can be represented as perfect squares. Beiler [4] gives one example of such triangles, namely; (63, 16, 65). Yiu [82] lists first few Pythagorean triangles with above stated property, namely;(63, 16, 65), (275, 252, 373), (735, 1088, 1313), (1539, 3100, 3461), (2783, 7056, 7585), (9075, 4508, 10133), (4563, 13916, 14645), (14703, 10496, 18065), (6975, 24832, 25793), (26195, 2268,

26293), (10115, 41148, 42373), (32079, 35600, 47921) and (14079, 64400, 65921).

But we found that the Pythagorean triangle (56355, 22748, 60773) has the above stated property and is not in the above list. This motivated us to examine the existence of all the Pythagorean Triangles, with their perimeters to be perfect squares. The matter of finding patterns of Special Pythagorean Triangles, with their perimeters as perfect square, is the prime concern of this chapter.

2.9 METHOD OF ANALYSIS

As seen before, the primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2, \qquad \textbf{(2.9.1)}$$

is given by Niven & Zuckerman [60] where,

$$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2, \qquad \textbf{(2.9.2)}$$

for some integers m, n of opposite parity such that $m > n > 0$ and

$(m, n) = 1$.

If the perimeter is a perfect square then for some $\beta \in \mathbf{N}$, we have

$$X + Y + Z = \beta^2. \qquad \textbf{(2.9.3)}$$

This by equation (2.9.2) gives perimeter P as

$$P = 2m\,(m+ n). \qquad \textbf{(2.9.4)}$$

Making the transformation,

$m = 2p^2$,

$n = q^2 - 2p^2$,

$$\beta = 2pq, \qquad \textbf{(2.9.5)}$$

where p and q are real numbers and

$$\sqrt{2}p \le q < 2p. \qquad \textbf{(2.9.6)}$$

From equation (2.9.4), we get

$P = 4p^2q^2 = \beta^2$.

*This paper is published in ANTARCTICA JOURNAL OF MATHEMATICS; 8(5); 2011;421-425.

To satisfy equation (2.9.2) only those values of p and hence q are taken for which m and n are positive integers.

Then, solutions of equation (2.9.1) in accordance with (2.9.3) are given by equation (2.9.5) and hence by equation (2.9.6) as follows:-

$X = q^2 (4p^2 - q^2)$,

$Y = 4p^2 (q^2 - 2p^2)$,

$Z = 8p^4 - 4p^2 q^2 + q^4$. **(2.9.7)**

Few examples of Primitive Pythagorean Triangles (X, Y, Z) with their perimeters as perfect squares are given in Table 2c.1 below:

Table 2c.1: Primitive Pythagorean Triangles (X, Y, Z)

S.N.	p	q	m	n	B	X	Y	Z	X^2	Y^2
1	2	3	8	1	12	63	16	65	3969	256
2	3	5	18	7	30	275	252	373	75625	63504
3	4	7	32	17	56	735	1088	1313	540225	1183744
4	5	9	50	31	90	1539	3100	3461	2368521	9610000
5	6	11	72	49	132	2783	7056	7585	7745089	49787136
6	7	13	98	71	182	4563	13916	14645	20820969	193655056
7	11	17	242	47	374	56355	22748	60773	3175886025	517471504

The following Table 2c.2 gives verification of these special Pythagorean Triangles.

Table 2c.2: $X^2 + Y^2 = Z^2$ and $X + Y + Z = \beta^2$

S.N.	$X^2 + Y^2$	Z^2	X + Y + Z	β^2
1	4225	4225	144	$144 = 12^2$
2	139129	139129	900	$900 = 30^2$
3	1723969	1723969	3136	$3136 = 56^2$
4	11978521	11978521	8100	$8100 = 90^2$
5	57532225	57532225	17424	$17424 = 132^2$
6	214476025	214476025	33124	$33124 = 182^2$
7	3693357529	3693357529	139876	$139876 = 374^2$

2.10 3D PLOTS

Solving the Diophantine equation,

$2m^2 + 2mn = \beta^2$,

i.e., $X + Y + Z = \beta^2$,

with the help of software Mathematica for $n < m$; $0 < m < 10^{18}$; $0 < n < 10^{18}$;

$0 < \beta < 10000$, we get only 352 Special Pythagorean Triangles (X, Y, Z) with their perimeters as perfect squares, and are primitive. Some of them, not given above are {44403, 57596, 72725}, {18963, 96316, 98165}, {82943, 576, 82945}, \

{77615, 42048, 88273}, {24863, 138816, 141025}, {113715, 15548, \

114773}, {103635, 69628, 124853}, {77763, 129116, 150725}, {31875, \

194012, 196613}, {134895, 107408, 172433}, {99375, 182672, 207953}, \

{40095, 264208, 267233}, {196259, 71100, 208741}, {49619, 351900, \

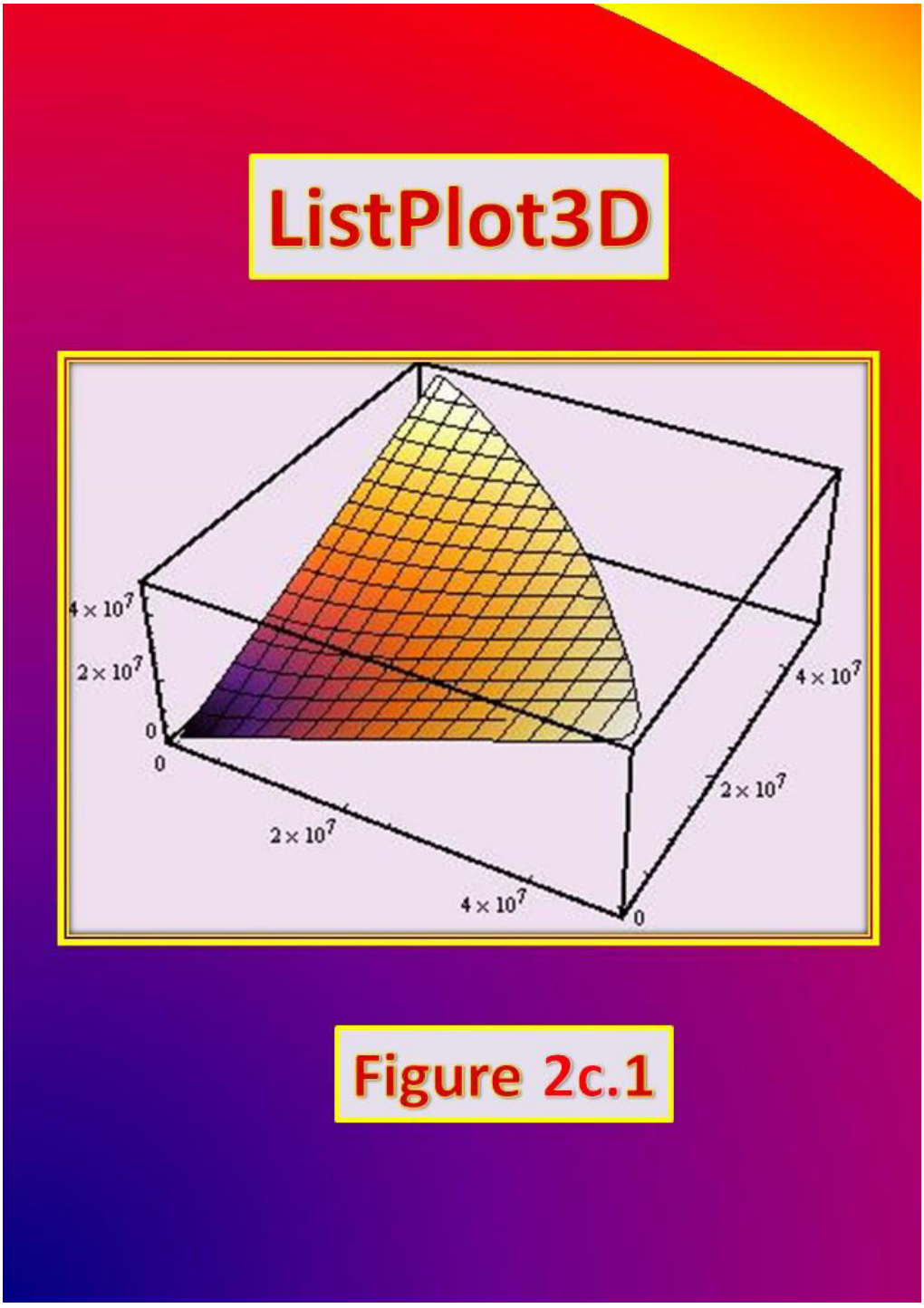
355381}, {261855, 17408, 262433}, {249375, 115712, 274913}, {215055, \

222208, 309233}, {153903, 336896, 370385}, {60543, 459776, 463745}, \

{331875, 54332, 336293}, {311283, 174556, 356885}, {264915, 304028, \

403253}, {187395, 442748, 480773}, {72963, 590716, 595205}.

The graphs of ListPlot3D (Figure 2c.1), ListSurfacePlot3D (Figure 2c.2) and ListPointPlot3D (Figure 2c.3) of these Special Pythagorean Triangles with their perimeters as perfect squares are given below: -



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ListSurfacePlot3D

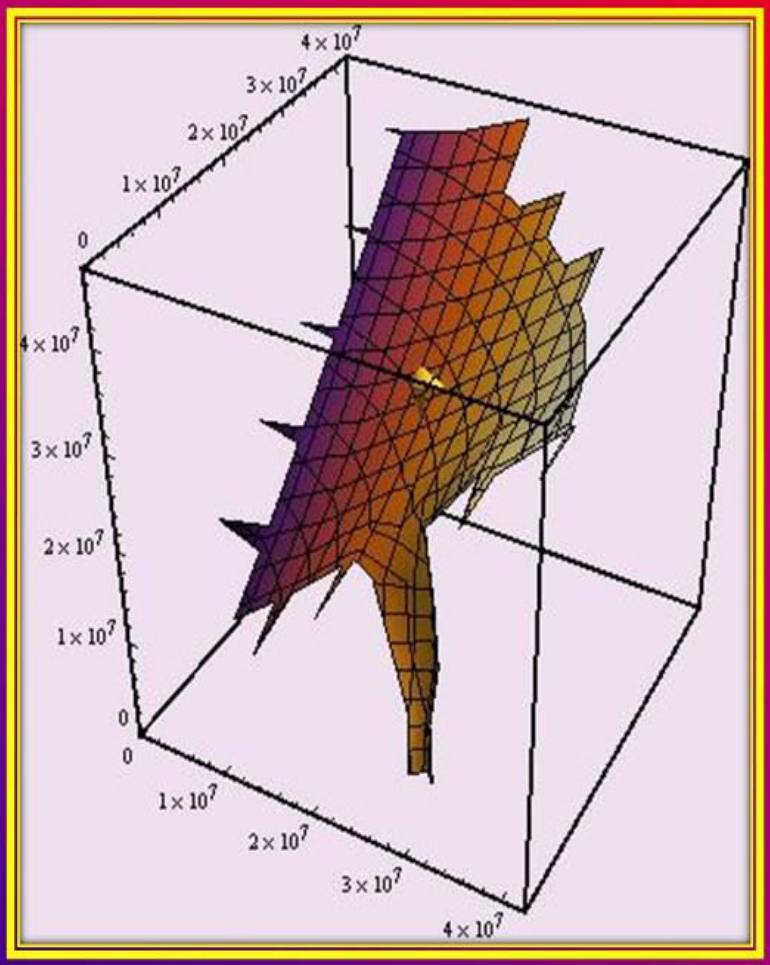

Figure 2c.2

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ListPointPlot3D

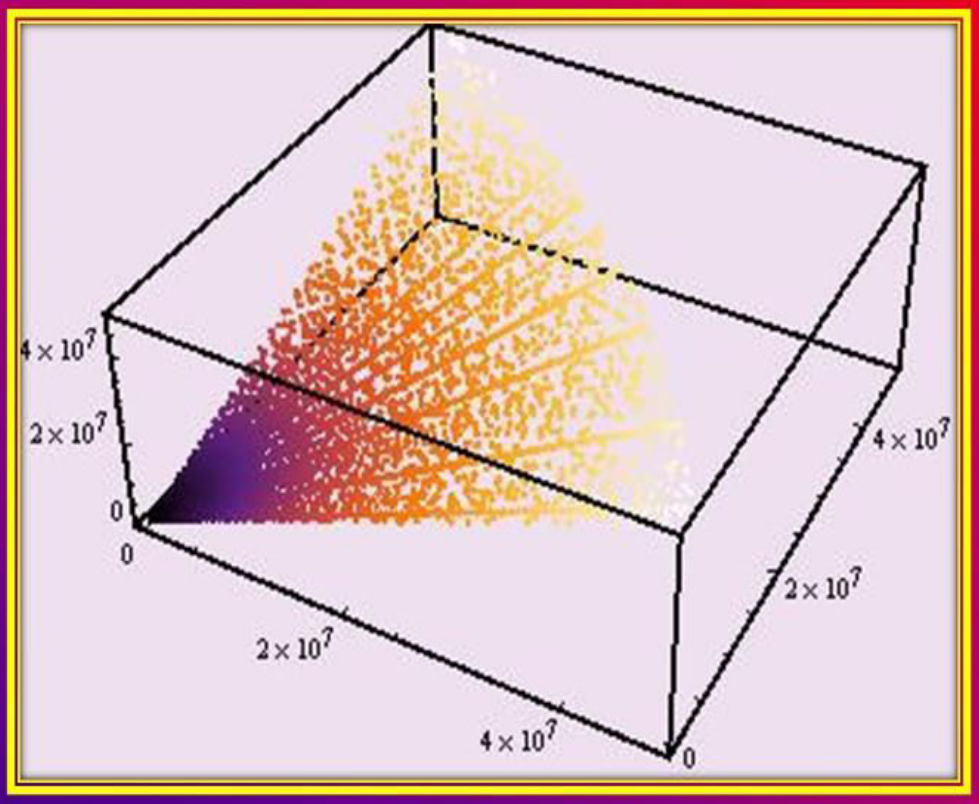

Figure 2c.3

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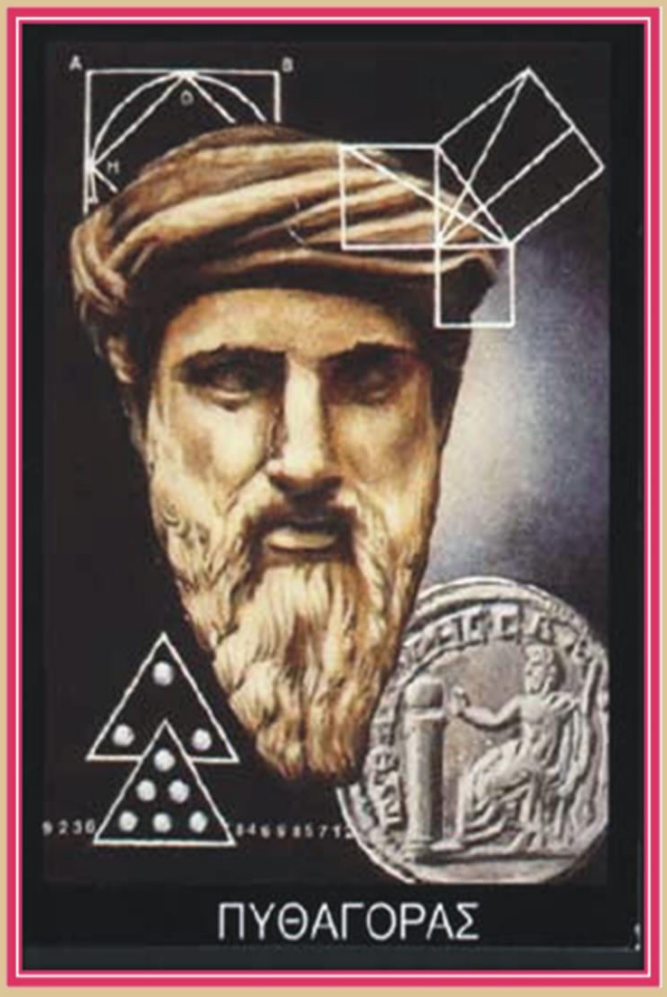
2.11 OBSERVATIONS

We also observe that

1. $X + Y + Z = (2pq)^2 = 0(\text{mod } 2)$.
2. $(Y + Z - X)^2 = 2(Y + Z)(Z - X) = 4q^4 (2p^2 - q^2)^2$.
3. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z) = 64\, p^4 (q^2 - p^2)^2$.

ISBN: 978-93-93810-76-2

CHAPTER THREE



SPECIAL PYTHAGOREAN TRIANGLES IN TERMS OF CUBIC AND QUARTIC DIOPHANTINE EQUATIONS

CHAPTER 3(A)

SPECIAL PYTHAGOREAN TRIANGLES WITH PERIMETER AS A SUM OF TWO SQUARES AND A CUBE*

3.1 INTRODUCTION

The integral solutions of cubic Diophantine equations are studied extensively by Gopalan and Premlatha [24], Janaki and Gopalan [51], Gopalan, Somnath and Vanitha [29], Pandichelvi [61], Gopalan and Pandichelvi [22], Gopalan and Vijyasankar [34], Gopalan and Krishnamoorthy [20]. The work of these mathematicians made us wonder whether there exists any Pythagorean Triangle whose sum of its legs is a cube. In this chapter, we make an attempt to find patterns of Special Pythagorean Triangles with perimeter as sum of two squares and a cube. A special case when two sides are consecutive is also obtained.

3.2 METHOD OF ANALYSIS

As we have seen in previous chapters, the primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2, \qquad \textbf{(3.2.1)}$$

is given by Niven & Zuckerman [60] where

$X = m^2 - n^2$,

$Y = 2mn$,

$$Z = m^2 + n^2, \qquad \textbf{(3.2.2)}$$

for some integers m, n of opposite parity such that $m > n > 0$ and

$(m, n) = 1$.

***Sum of two legs is a cube:** If the sum of two legs of the Pythagorean Triangle (X, Y, Z) is a cube, i.e.,

$$X + Y = \beta^3, \beta \in \mathbf{N}, \qquad \textbf{(3.2.3)}$$

which, by virtue of equation (3.2.2), becomes

$$m^2 - n^2 + 2mn = \beta^3. \qquad \textbf{(3.2.4)}$$

Then, by the theory of equations, solutions of equation (3.2.4) are given by

$\beta = (m^2 - n^2 + 2mn)^{1/3}$,

$\beta = - (-1)^{1/3} (m^2 - n^2 + 2mn)^{1/3}$,

$$\beta = (-1)^{2/3} (m^2 - n^2 + 2mn)^{1/3}. \qquad \textbf{(3.2.5)}$$

*** Perimeter is sum of two squares and a cube:** If the sum of two legs is a cube, then by equation (3.2.3),

$X + Y = \beta^3$.

And by equation (3.2.2),

$Z = m^2 + n^2$.

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* This Paper is published in International Journal of Mathematical Sciences and Applications;1(3); Sept.2011; **1225-28.**

Adding these equations we get,

$X + Y + Z = \beta^3 + m^2 + n^2$. **(3.2.6)**

Equations (3.2.2) and (3.2.5) generates X and Y which satisfy equations (3.2.1) and (3.2.3) in correspondence with equation (3.2.6). Few examples of Pythagorean triangles (X, Y, Z), not all of them primitives, where

$X + Y + Z = \beta^3 + m^2 + n^2$, are given in the Table 3a.1 below:

Table 3a.1: Special Pythagorean Triangles

S.N.	m	n	X	Y	Z	β	$X + Y = \beta^3$
1	14	7	147	196	245	7	$147 + 196 = 343 = 7^3$
2	16	3	247	96	265	7	$247 + 96 = 343 = 7^3$
3	51	34	1445	3468	3757	17	$1445 + 3468 = 4913 = 17^3$
4	63	8	3905	1008	4033	17	$3905 + 1008 = 4913 = 17^3$
5	78	77	155	12012	12013	23	$155 + 12012 = 12167 = 23^3$
6	92	23	7935	4232	8993	23	$7935 + 4232 = 12167 = 23^3$

3.3 OBSERVATIONS:

We observe that

3. For every value of β, there are two pairs of m and n.
4. As given in row 10, with β given, m and n can be found out.
5. Considering the special case as in row 1, 4, 8, …, we find that n = β.

***n = β:** For this special case,

Putting n = β in equation (3.2.4), we get

$m^2 - n^2 + 2mn = n^3$.

This gives

$m = - n \pm \sqrt{(2n^2 + n^3)}$.

As $\sqrt{(2 n^2 + n^3)} > n$, we discard $m = - n - \sqrt{(2n^2 + n^3)}$.

Then, $m = - n + \sqrt{(2n^2 + n^3)}$. **(3.3.1)**

To solve equation (3.3.1), we make the substitution,

$n = p^2 - 2$, **(3.3.2)**

where p is a positive integer greater than 2.

From equations (3.3.1) and (3.3.2), we get

$m = (p^2 - 2)(p - 1)$. **(3.3.3)**

Therefore, from equations (3.2.2), (3.3.2) and (3.3.3), we get

$X = (p^2 - 2)^2 (p^2 - 2p)$,

$Y = 2(p^2 - 2)^2 (p - 1)$,

$Z = (p^2 - 2)^2 (p^2 - 2p + 2)$,

$X + Y = (p^2 - 2)^3 = \beta^3$. **(3.3.4)**

From equation (3.3.4), we get

$X + Y + Z = (p^2 - 2)^3 + (p^2 - 2)^2(p - 1)^2 + (p^2 - 2)^2$ **(3.3.5)**

Equation (3.3.4) generates Pythagorean triangles (X, Y, Z) which satisfy equations (3.2.1) and (3.2.3) in correspondence with equations (3.3.2) and (3.3.5).

Few examples of Pythagorean triangles (X, Y, Z), where, $X + Y = n^3$,

$X + Y + Z = n^3 + m^2 + n^2$, are given in the Table 3a.2 below. Table 3a.3 gives the verification of these Pythagorean triangles.

Table 3a.2: Generation of (X, Y, Z) with $n = \beta$; $X + Y + Z = \beta^3 + m^2 + n^2$

S.N.	P	m	n	B	X	Y	Z
1	3	14	7	7	147	196	245
2	5	92	23	23	7935	4232	8993
3	7	282	47	47	77315	26508	81733
4	9	632	79	79	393183	100488	405665
5	11	1190	119	119	1401939	283220	1430261

Table 3a.3: Verification of $X + Y = n^3$ and $X + Y + Z = n^3 + m^2 + n^2$

S.N.	$X + Y = n^3$	$X + Y + Z = n^3 + m^2 + n^2$
1	$147 + 196 = 7^3$	$588 = 7^3 + 14^2 + 7^2$
2	$7935 + 4232 = 23^3$	$21160 = 23^3 + 92^2 + 23^2$
3	$77315 + 26508 = 103823 = 47^3$	$185556 = 47^3 + 282^2 + 43^2$
4	$393183 + 100488 = 493671 = 79^3$	$899336 = 79^3 + 632^2 + 79^2$
5	$1401939 + 283220 = 1685159 = 119^3$	$3115420 = 119^3 + 1190^2 + 119^2$

*** One Leg and a Side are Consecutive:**

As we have seen in Chapter 2A, in such cases, $m = n + 1$, then equation (3.2.4) becomes $2n^2 + 4n + 1 - \beta^3 = 0$.

Using software Mathematica, this Diophantine equation was solved for

$0 < n < 10^{11}$ & $0 < \beta < 10^4$ and it was found that there was just one solution for n and β, namely, $n = 77$, $\beta = 23$.

Therefore, $m = n + 1 \Rightarrow$ □ □ □ □ $m = 78$, which by equation (3.2.2), gives

$X = 155$,

$Y = 12012$,

$Z = 12013$.

Therefore,

$X^2 + Y^2 = Z^2$

$\Rightarrow 155^2 + 12012^2 = 12013^2$

$\Rightarrow 24025 + 144288144 = 144312169$.

And

$X + Y + Z = 155 + 12012 = 12167 = 23^3 = \beta^3$.

For $n \geq 10^{11}$ the existence of special Pythagorean triangles with two sides consecutive and perimeter to be sum of two squares and a cube needs to be investigated.

Solving the Diophantine equation, $m^2 + 2mn + n^2 - \beta^3 = 0$, when $n < m$; $0 < m < 10^5$; $0 < n < 10^5$; $0 < \beta < 10^5$, we get 1993 solutions of (m, n, β) out of which only 319 are primitives. Some

of the special Pythagorean Triangles (X, Y, Z), not necessarily primitive, where sum of legs is a cube and perimeter is sum of two squares and a cube are listed below with values less than 10^7: (147, 196, 245), (247, 96, 265), (768, 1976, 2120), (1568, 1176, 1960), (1445, 3468, 3757),(3905,1008,4033), (155, 12012, 12013), (7935, 4232, 8993), (9408, 12544, 15680), (15808, 6144, 16960), (6727, 23064, 24025), (17751, 12040, 21449), (8064, 31240, 32264), (27744, 11560, 30056), (31937, 36984, 48865), (35301, 33620, 48749), (33856, 63480,71944), (96096, 1240, 96104), (33495, 70328, 77897), (77315, 26508, 81733), (21609, 96040, 98441), (57229, 60420, 83221), (77469, 40180, 87269), (49152, 126464, 135680), (100352, 75264, 125440), (96320, 142008, 171592), (184512, 53816, 192200), (107163, 142884, 178605), (180063, 69984, 193185), (92480, 221952, 240448), (249920, 64512, 258112), (55451, 302460, 307501), (148211, 209700, 256789), (48793, 340224, 343705), (239805, 149212, 282437), (229419, 263620, 349469), (393183, 99856, 405665), (268960, 282408, 389992), (295872, 255496, 390920), (261393, 443576, 514865), (650093, 54876, 652405), (9920, 768768, 768832), (507840, 270848, 575552), (212064, 618520, 653864), (562624, 267960, 623176), (122317, 790356, 799765), (330105, 582568, 669593), (321440, 619752, 698152), (483360, 457832, 665768), (768320, 172872, 787528), (434283, 658444, 788765), (583495, 509232, 774457).

3.4 3D PLOTS:

Plotting 1993 solutions which are obtained above as ListPlot3D (Figure 3a.1), ListSurfacePlot3D (Figure 3a.2) and ListPointPlot3D (Figure 3a.3), we get following figures:

ISBN: 978-93-93810-76-2

ListPlot3D

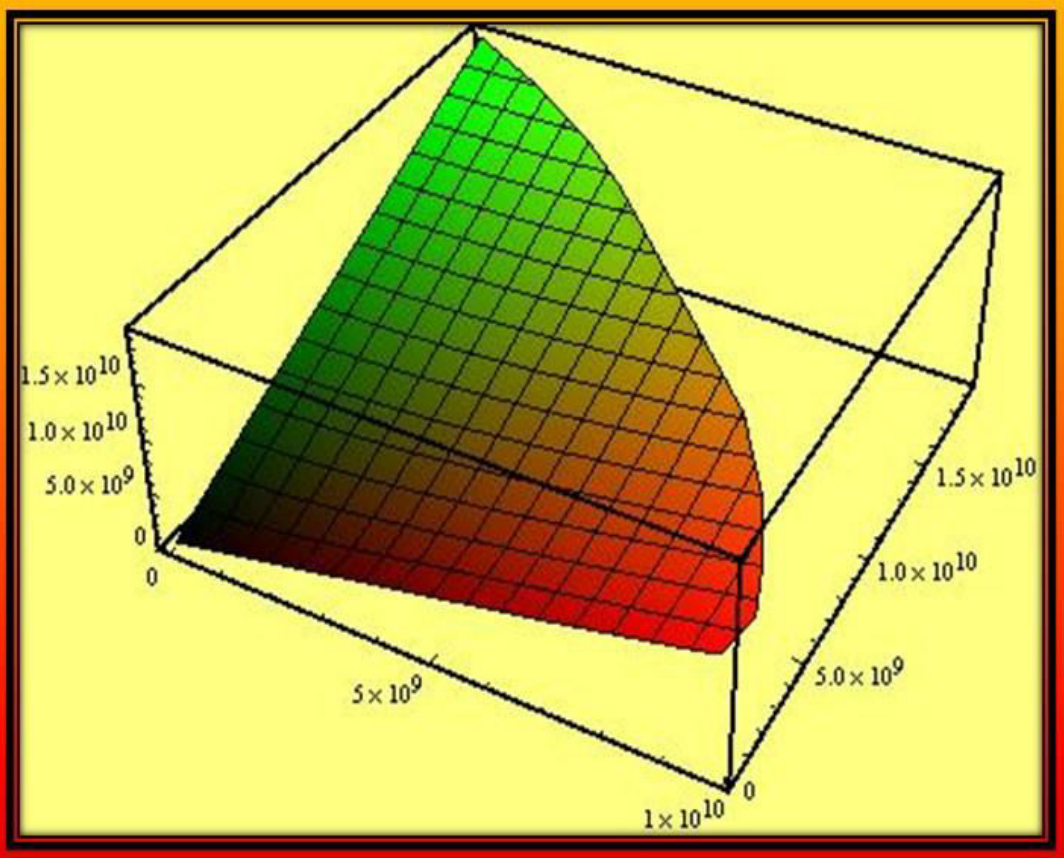
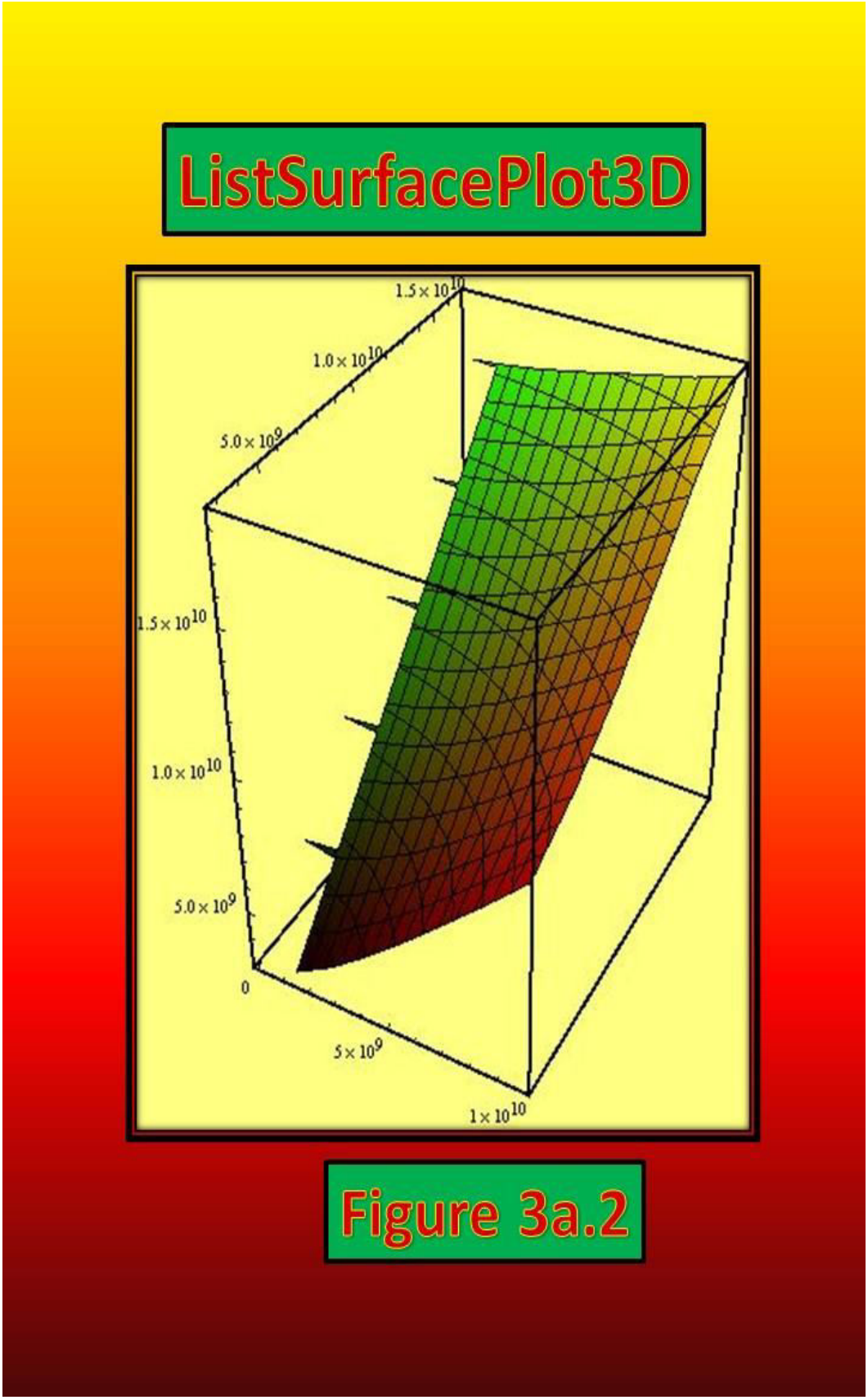


Figure 3a.1

ISBN: 978-93-93810-76-2



ISBN: 978-93-93810-76-2

ListPointPlot3D

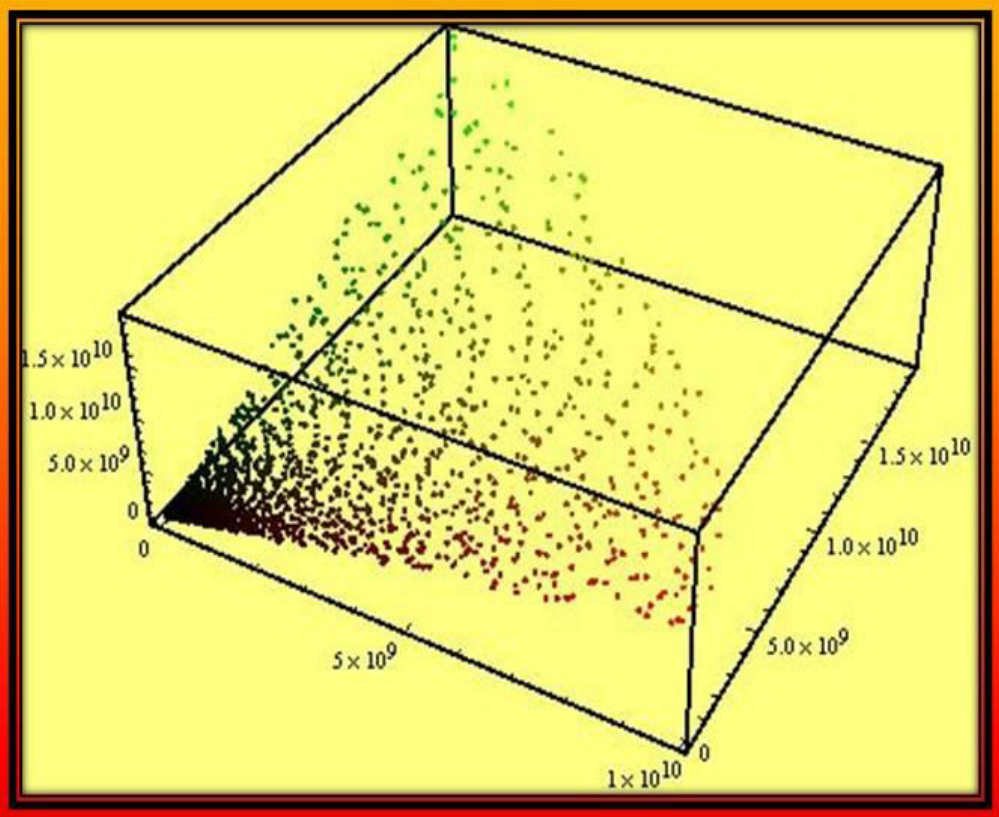


Figure 3a.3

CHAPTER 3(B)

SPECIAL PYTHAGOREAN TRIANGLES WITH PERIMETER TO BE SUM TWO SQUARES AND A QUARTIC*

3.5 INTRODUCTION

After finding special Pythagorean Triangles, each of whose perimeters are sum of two squares and a cube, in the previous chapter, we tried to extend the result for quartic Diophantine equation. Gopalan and Sangeetha [25], Gopalan and Padma [23], Janaki and Gopalan [52], Gopalan and Pandichelvi [21], Gopalan and Shanmuganandham [26] have discussed the solutions of quartic Diophantine equations. Some such Special Pythagorean Triangles with perimeter as sum of two squares and a quartic are sought for in this chapter. Various 3D graphs of these are also plotted.

3.6 METHOD OF ANALYSIS

We know that, the primitive solutions of the Pythagorean Equation

$$X^2 + Y^2 = Z^2 \qquad \textbf{(3.6.1)}$$

is given by Niven and Zuckerman [60] where

$$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2 \qquad \textbf{(3.6.2)}$$

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

***Sum of two legs is a Quartic:** If in (X, Y, Z), the sum of two legs is a quartic, then for some $\beta \in \mathbf{N}$, $X + Y = \beta^4$ **(3.6.3)**

$$\text{i.e., } m^2 + 2mn - n^2 - \beta^4 = 0 \qquad \textbf{(3.6.4)}$$

Then, by theory of equations, solutions of (3.6.4) and subsequently (3.6.3) are given by,

$\beta = -(m^2 + 2mn - n^2)^{1/4}$

$\beta = -i(m^2 + 2mn - n^2)^{1/4}$

$\beta = (m^2 + 2mn - n^2)^{1/4}$

$\beta = i(m^2 + 2mn - n^2)^{1/4}$ **(3.6.5)**

From these values only those values are taken for which β is a positive integer.

***Perimeter is sum of two squares and a quartic:**

From Equations (3.4.2) and (3.4.3),

$$X + Y + Z = \beta^4 + m^2 + n^2. \qquad \textbf{(3.6.6)}$$

Equations (3.6.2) and (3.6.4) generate X and Y which satisfy (3.6.1) and (3.6.3) in correspondence with (3.6.6).

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* This Paper has been published in International Journal of Mathematical Archive; 2(7); (July 2011); 1094-98.

Solving the Diophantine equation (3.6.4), $m^2 + 2\,m\,n - n^2 - \beta^4 = 0$, using software Mathematica, for $n < m$; $0 < n < 10^6$; $0 < m < 10^6$;

$0 < \beta < 10^6$, we get the 1698 values for (X, Y, Z) which will consume many pages to write down! Out of these 1698 Pythagorean triangles, each of whose perimeters is a sum of two distinct squares and a quartic, only 141 are Primitives! Following are the few examples (precisely 74) with $0 < \beta < 10^2$ given in Table 3b.1:

Table 3b.1: Special Pythagorean Triangles (X, Y, Z)

S.N.	m	n	β	$X = m^2 - n^2$	$Y = 2mn$	$Z = m^2 + n^2$
1	35	28	7	441	1960	2009
2	41	10	7	1581	820	1781
3	205	182	17	8901	74620	75149
4	221	102	17	38437	45084	59245
5	315	252	21	35721	158760	162729
6	369	90	21	128061	66420	144261
7	389	238	23	94677	185164	207965
8	391	230	23	99981	179860	205781
9	689	528	31	195937	727584	753505
10	775	248	31	539121	384400	662129
11	875	700	35	275625	1225000	1255625
12	1025	250	35	988125	512500	1113125
13	1189	1148	41	95817	2729944	2731625
14	1625	58	41	2637261	188500	2643989
15	1565	1428	47	410041	4469640	4488409
16	1739	658	47	2591157	2288524	3457085
17	1715	1372	49	1058841	4705960	4823609
18	1781	1020	49	2131561	3633240	4212361
19	1855	798	49	2804221	2960580	4077829
20	2009	490	49	3795981	1968820	4276181
21	1845	1638	51	720981	6044220	6087069
22	1989	918	51	3113397	3651804	4798845
23	2835	2268	63	2893401	12859560	13181049
24	3321	810	63	10372941	5380020	11685141
25	3501	2142	69	7668837	14998284	16845165
26	3519	2070	69	8098461	14568660	16668261
27	3865	1752	71	11868721	13542960	18007729
28	4331	852	71	18031657	7380024	19483465
29	3869	2628	73	8062777	20335464	21875545
30	4105	1802	73	13603821	14794420	20098229
31	4235	3388	77	6456681	28696360	29413769
32	4961	1210	77	23147421	12005620	26075621
33	4549	2988	79	11765257	27184824	29621545
34	5135	1422	79	24346141	14603940	28390309
35	5125	4550	85	5563125	46637500	46968125
36	5525	2550	85	24023125	28177500	37028125
37	5785	3738	89	19493581	43248660	47438869
38	5989	2990	89	26928021	35814220	44808221
39	5915	4732	91	12595401	55979560	57379049
40	6929	1690	91	45154941	23420020	50867141

41	6201	4752	93	15870897	58934304	61033905
42	6975	2232	93	43668801	31136400	53632449
43	7421	2772	97	47387257	41142024	62755225
44	8245	1358	97	66135861	22393420	69824189

Following Table 3b.2 verifies the result for first eight special Pythagorean triangles-

Table 3b.2: $X^2 + Y^2 = Z^2$; $X + Y = \beta^4$; $X + Y + Z = \beta^4 + m^2 + n^2$

S.N.	X^2	Y^2	$X^2 + Y^2$	Z^2	$X+Y=\beta^4$	$X + Y + Z = \beta^4 + m^2 + n^2$
1	**194481**	3841600	4036081	4036081	$2401=7^4$	$4410=7^4+35^2+28^2$
2	**2499561**	672400	3171961	3171961	$2401=7^4$	$4182=7^4+41^2+10^2$
3	**79227801**	5568144400	5647372201	5647372201	$83521=17^4$	$158670=17^4+205^2+182^2$
4	**1477402969**	2032567056	3509970025	3509970025	$83521=17^4$	$142766=17^4+221^2+102^2$
5	**1275989841**	25204737600	26480727441	26480727441	$194481=21^4$	$357210=21^4+315^2+252^2$
6	**1639961721**	4411616400	20811236121	20811236121	$194481=21^4$	$338742=21^4+369^2+90^2$
7	**8963734329**	34285706896	43249441225	43249441225	$279841=23^4$	$487806=23^4+389^2+238^2$
8	**9996200361**	32349619600	42345819961	42345819961	$279841=23^4$	$485622=23^4+391^2+230^2$

***One Leg and Hypotenuse are Consecutive:**

From the equation (2.2.7) of chapter 2A, in such cases, m = n + 1.

Substituting the value of m in equations (3.6.2) and (3.6.4), we get the Diophantine equation,

$2n^2 + 4n+1- \beta^4 = 0.$ **(3.6.7)**

Using the software Mathematica equation (3.6.7) was solved for

$0 < n < 10^7$ and $0 < \beta < 10^7$ and it was found that it has no solution. It needs further investigations to find whether the solution of (3.6.7) exists for $n \geq 10^7$.

3.7 3D PLOTS

For $n < m$, $0 < n < 10^6$, $0 < m < 10^6$, $0 < \beta < 10^6$, we get the 1698 values for (X, Y, Z). Plotting these values as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D, we get the following Figure 3b.1, Figure 3b.2 and Figure 3b.3 respectively:-

ISBN: 978-93-93810-76-2

ListPlot3D

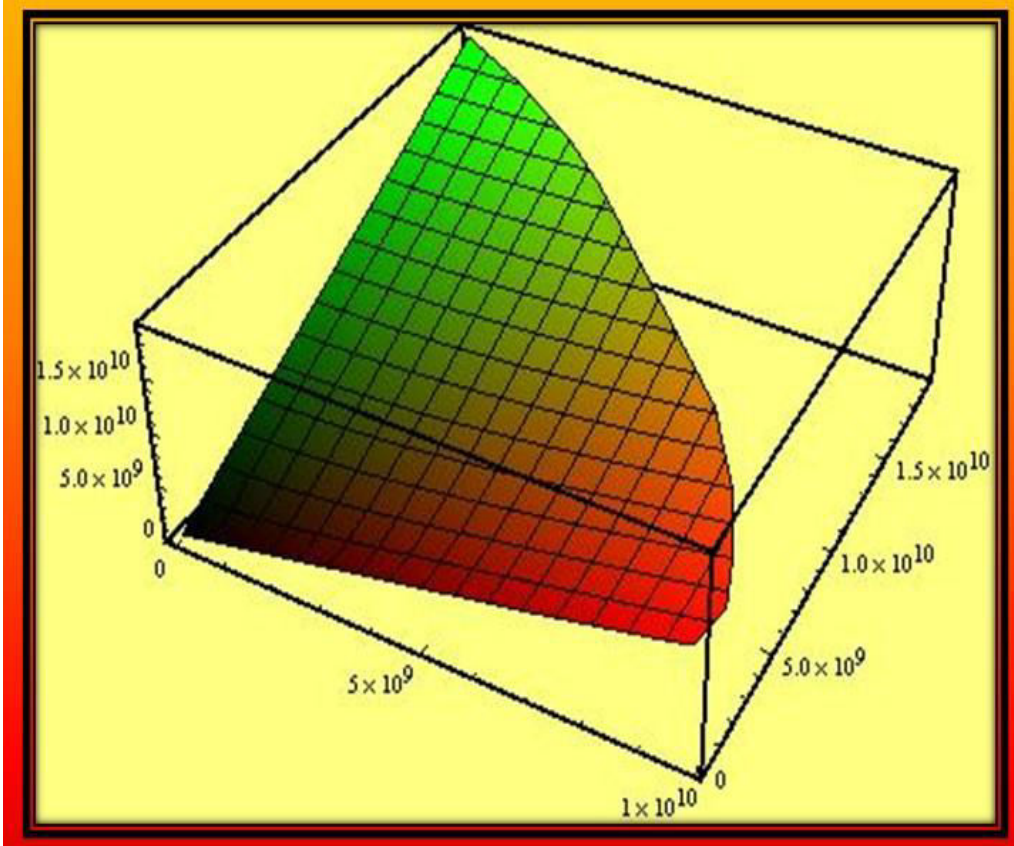

Figure 3a.1

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ListSurfacePlot3D

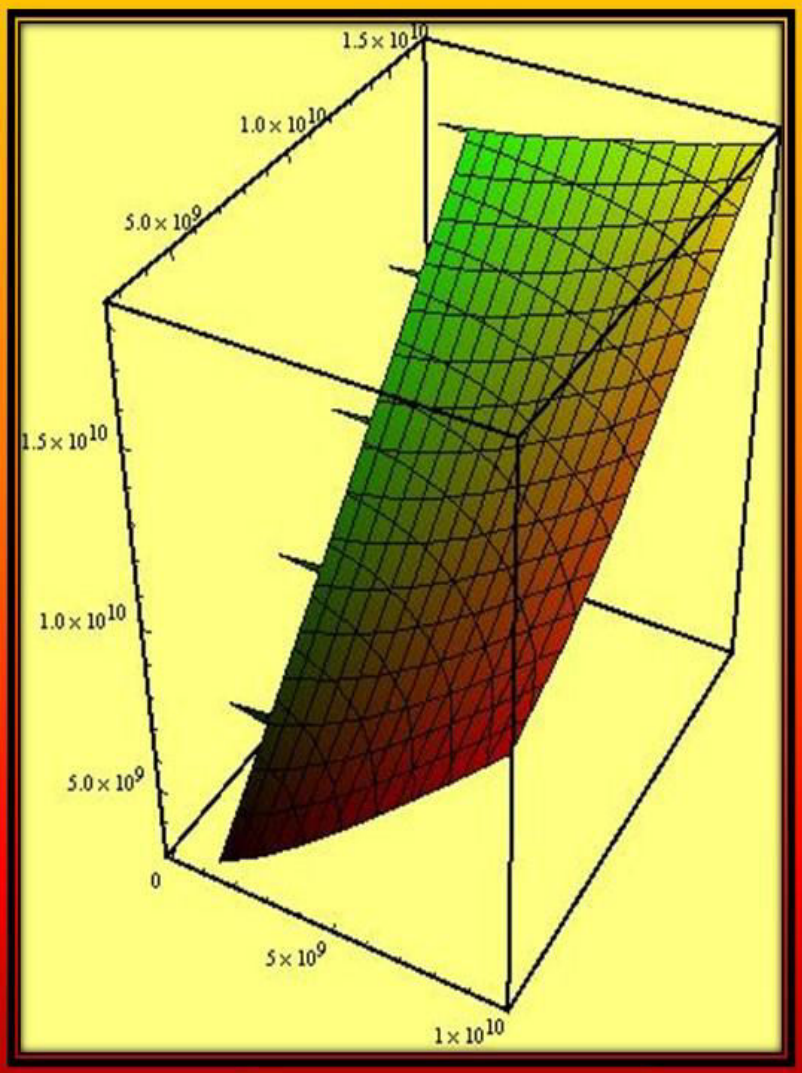

Figure 3a.2

ISBN: 978-93-93810-76-2

ListPointPlot3D

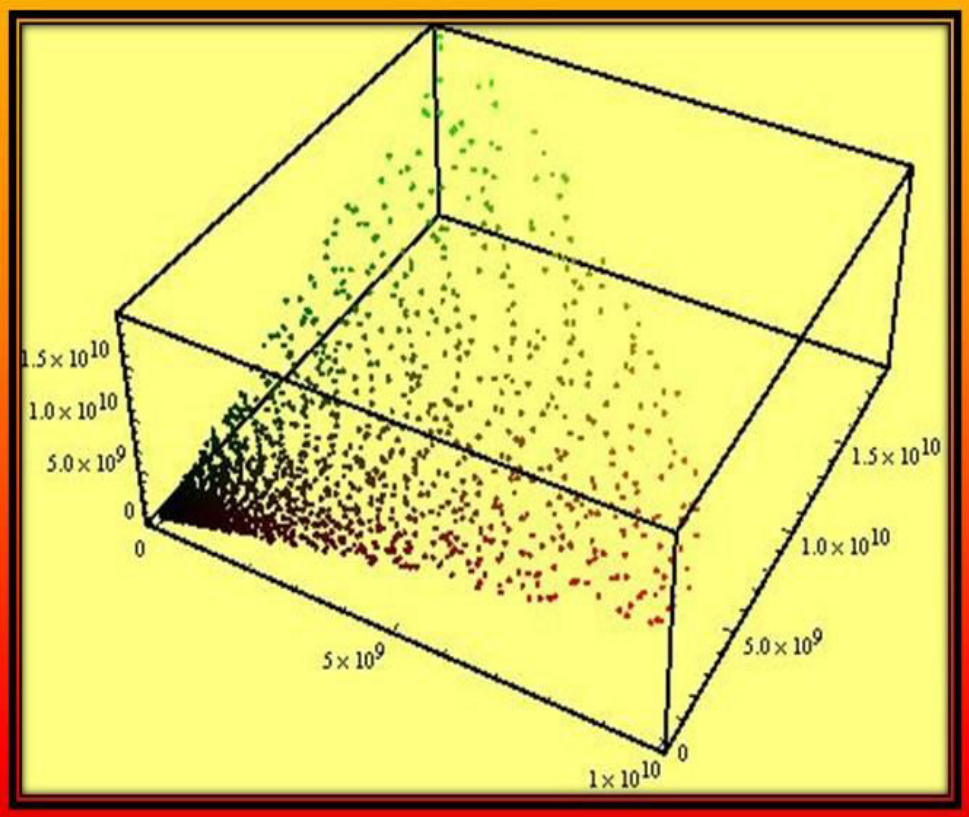

Figure 3a.3

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3.8 OBSERVATIONS

It is also observed that

1. Except for $\beta = 49$ and $\beta = 98$, for each $\beta < 100$, there are two pairs of m and n and hence two values of (X, Y, Z).
2. Four Pythagorean Triangles (X, Y, Z) correspond to $\beta = 49$ and $\beta = 98$.
3. $X + Y + Z = 0 (\text{mod } 2)$.
4. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.

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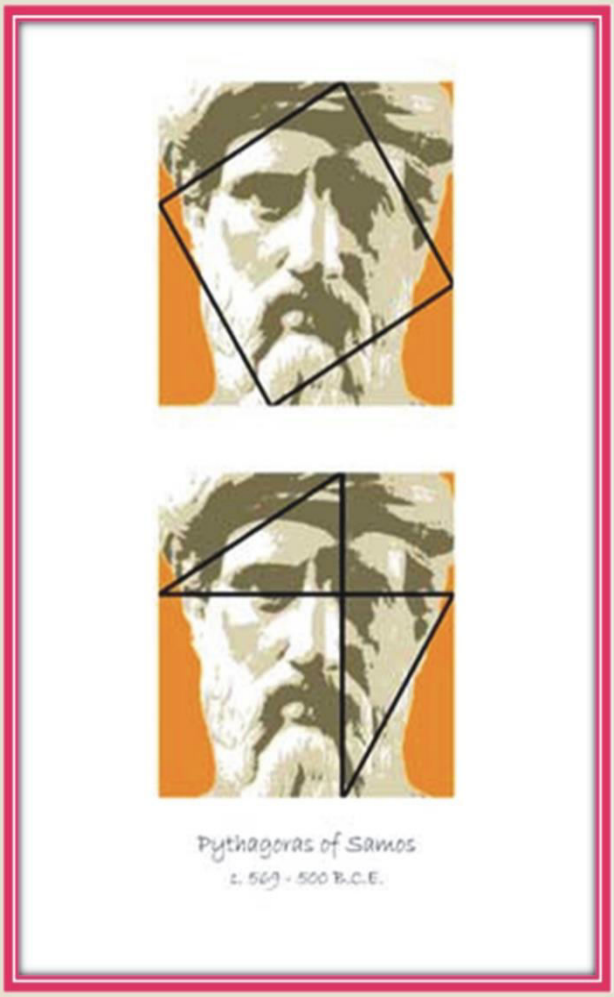
CHAPTER FOUR



SPECIAL PYTHAGOREAN TRIANGLES AS SOLUTIONS OF QUINTIC AND HEXIC DIOPHANTINE EQUATIONS

CHAPTER 4(A)

SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS A QUINTIC*

4.1 INTRODUCTION

Choudhary [8] has studied the solvability of quintic and sextic Diophantine equations. The determination of the Pythagorean triangles, with their perimeters as the sum of two distinct squares and a quartic, in the previous chapter, encouraged us to find the solution of the quintic Diophantine equation which gives a Pythagorean Triangle, with sum of its legs a quintic. The existence of such triangles, with two sides consecutive, is also investigated.

4.2 METHOD OF ANALYSIS

The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2, \tag{4.2.1}$$

$$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2, \tag{4.2.2}$$

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

*** Sum of two legs is a quintic:** If, in Pythagorean triangle, sum of two legs is a quintic, then

$$X + Y = \beta^5, \quad \text{for some } \beta \in N. \tag{4.2.3}$$

This, by equation (4.2.2), becomes,

$$m^2 + 2mn - n^2 - \beta^5 = 0. \tag{4.2.4}$$

Then, by theory of equations, solutions of (4.2.4) are given by,

$$\beta = (m^2 + 2mn - n^2)^{1/5},$$

$$\beta = -(-1)^{1/5}(m^2 + 2mn - n^2)^{1/5},$$

$$\beta = (-1)^{2/5}(m^2 + 2mn - n^2)^{1/5},$$

$$\beta = -(-1)^{3/5}(m^2 + 2mn - n^2)^{1/5},$$

$$\beta = (-1)^{4/5}(m^2 + 2mn - n^2)^{1/5}. \tag{4.2.5}$$

From these, only those values are taken for which β is a positive integer.

***Perimeter is sum of two squares and a quintic:**

From equations (4.2.2) and (4.2.3),

$$X + Y + Z = \beta^5 + m^2 + n^2. \tag{4.2.6}$$

(4.2.2) and (4.2.4) generate (X, Y, Z) which satisfy (4.2.1) and (4.2.3) in correspondence with (4.2.6).

Solving the Diophantine equation (4.2.4) by Mathematica,

$m^2 + 2mn - n^2 - \beta^5 = 0$, for $n < m$; $0 < n < 10^9$; $0 < m < 10^9$; $0 < \beta < 10^9$; we get the 7080 values for (m, n, β). These, in turn, will give 7080 values for (X, Y, Z) which will consume around 90 pages to write down! Out of these 7080 values only 549 are primitives.

Following are the first seven examples out of 74, not all primitives, with $0 < \beta < 25$, in Table 4a.1:-

*This Paper is published in International Journal: Mathematical Manuscripts (IJMM); (September 2011); 1-4.

Table 4a.1: Special Pythagorean Triangles (X, Y, Z)

S.N.	m	n	β	$X = m^2 - n^2$	$Y = 2mn$	$Z = m^2 + n^2$
1	92	81	7	1903	14904	15025
2	98	49	7	7203	9604	12005
3	112	21	7	12103	4704	12985
4	867	578	17	417605	1002252	1085773
5	979	274	17	883365	536492	1033517
6	1071	136	17	1128545	291312	1165537
7	1794	1771	23	81995	6354348	6354877

Table 4a.2 verifies that perimeters of these Pythagorean Triangles are sum of two squares and a quintic.

Table 4a.2: $X + Y = \beta^5$; $X^2 + Y^2 = Z^2$ & $X + Y + Z = \beta^5 + m^2 + n^2$

S.N.	X^2	Y^2	$X^2 + Y^2$	Z^2	$X + Y = \beta^5$	$X + Y + Z = \beta^5 + m^2 + n^2$
1	3621409	222129216	225750625	225750625	$16807 = 7^5$	$31832 = 7^5 + 92^2 + 81^2$
2	51883209	92236816	144120025	144120025	$16807 = 7^5$	$28812 = 7^5 + 98^2 + 49^2$
3	146482609	22127616	168610225	168610225	$16807 = 7^5$	$29792 = 7^5 + 112^2 + 21^2$
4	174393936025	1004509071504	1178903007529	1178903007529	$1419857 = 17^5$	$2505630 = 17^5 + 867^2 + 578^2$
5	780333723225	2878236660064	1068157389289	1068157389289	$1419857 = 17^5$	$2453374 = 17^5 + 979^2 + 274^2$
6	1273613817025	84862681344	1358476498369	1358476498369	$1419857 = 17^5$	$2585394 = 17^5 + 1071^2 + 136^2$
7	6723180025	40377738505104	4038446168585129	4038446168585129	$6436343 = 23^5$	$12791220 = 23^5 + 1794^2 + 1771^2$

***One Leg and Hypotenuse are Consecutive:**

We know that, from equation (2.2.7), in such cases, m = n + 1.

Substituting the value of m in equations (4.2.2) and (4.2.4), we get the Diophantine equation,

$2n^2 + 4n + 1 - \beta^5 = 0.$ **(4.2.7)**

Using the software Mathematica equation (4.2.7) was solved for

$0 < n < 10^8$; $0 < \beta < 10^8$ and it was found that it has no solution. It needs further investigations to find whether the solution of equation (4.2.7) exists for $n \geq 10^8$.

4.3 3D PLOTS

As seen above, for $n < m$; $0 < n < 10^9$; $0 < m < 10^9$;

$0 < \beta < 10^9$, we get 7080 values for (X, Y, Z). Plotting these values as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D using software Mathematica, we get following figures 4a.1, 4a.2 and 4a.3 respectively: -

ISBN: 978-93-93810-76-2

ListPlot3D

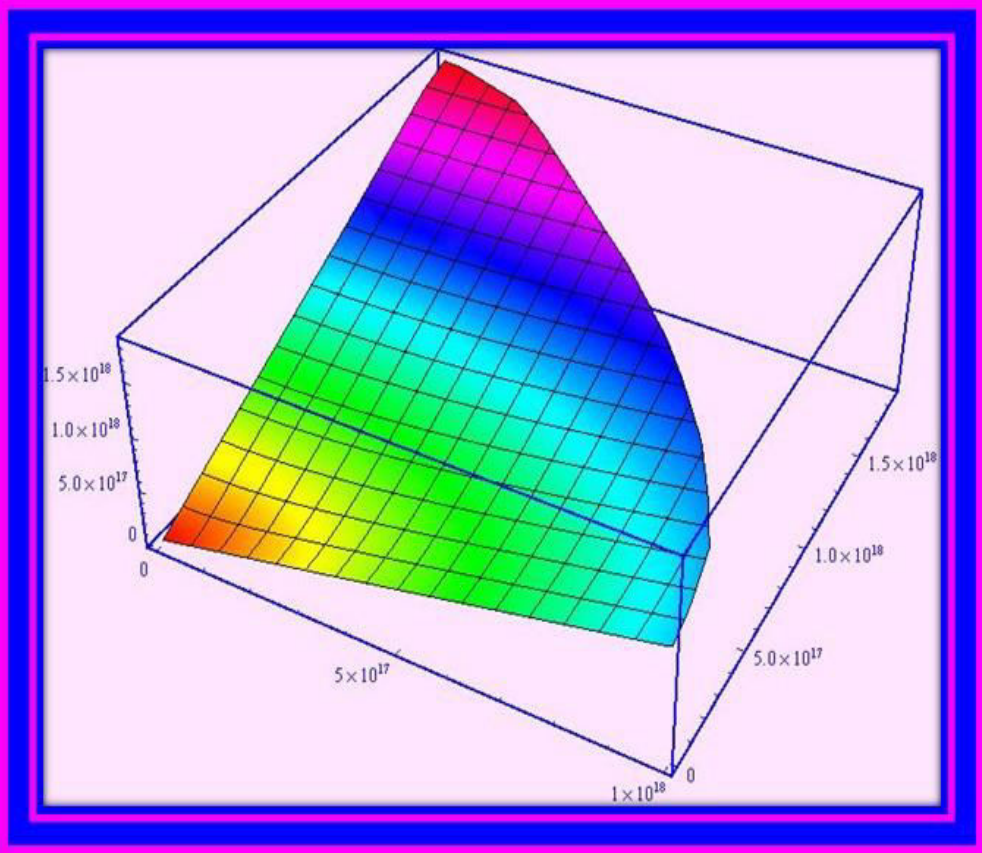


Figure 4a.1

ISBN: 978-93-93810-76-2

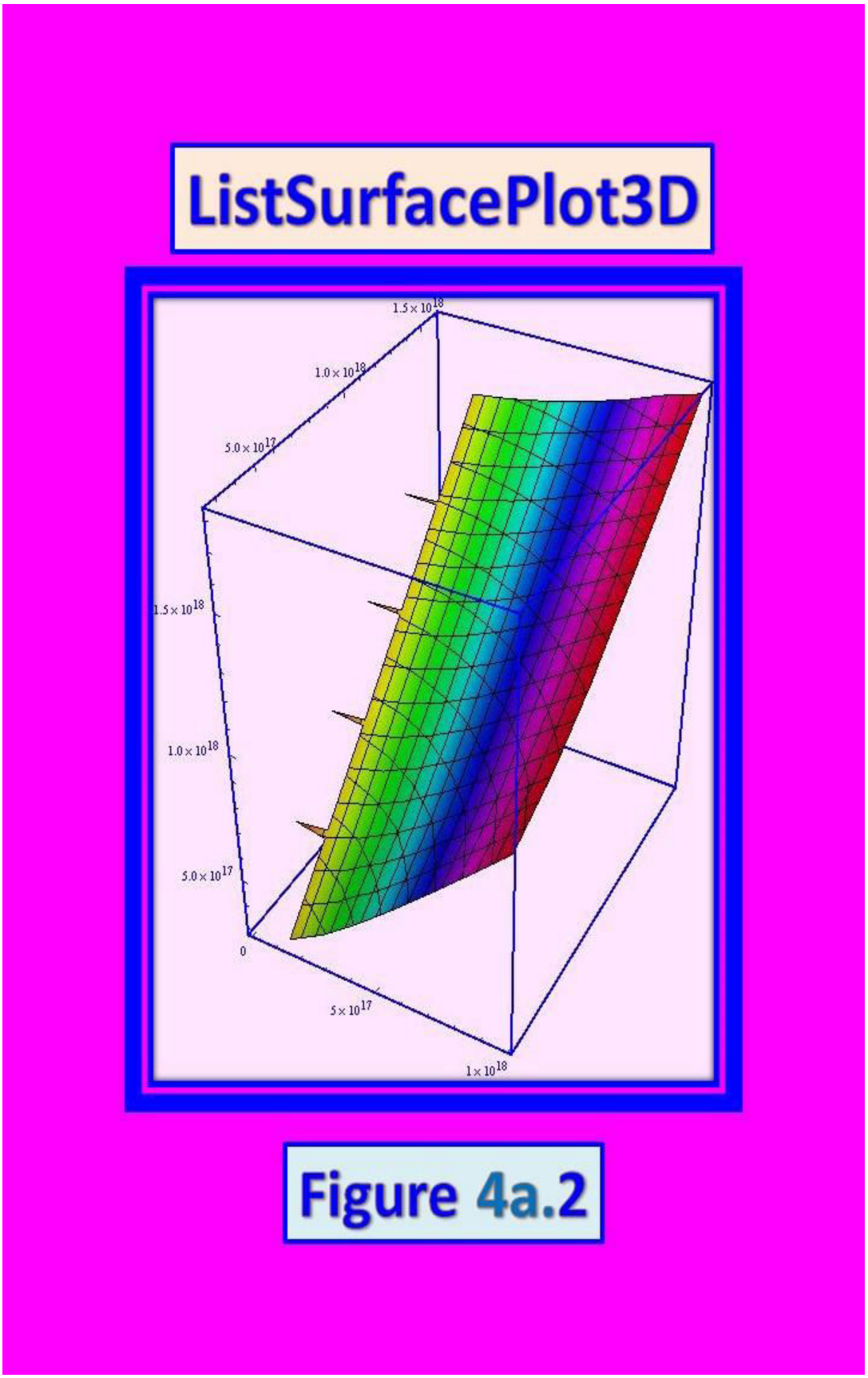


Figure 4a.2

ISBN: 978-93-93810-76-2

ListPointPlot3D

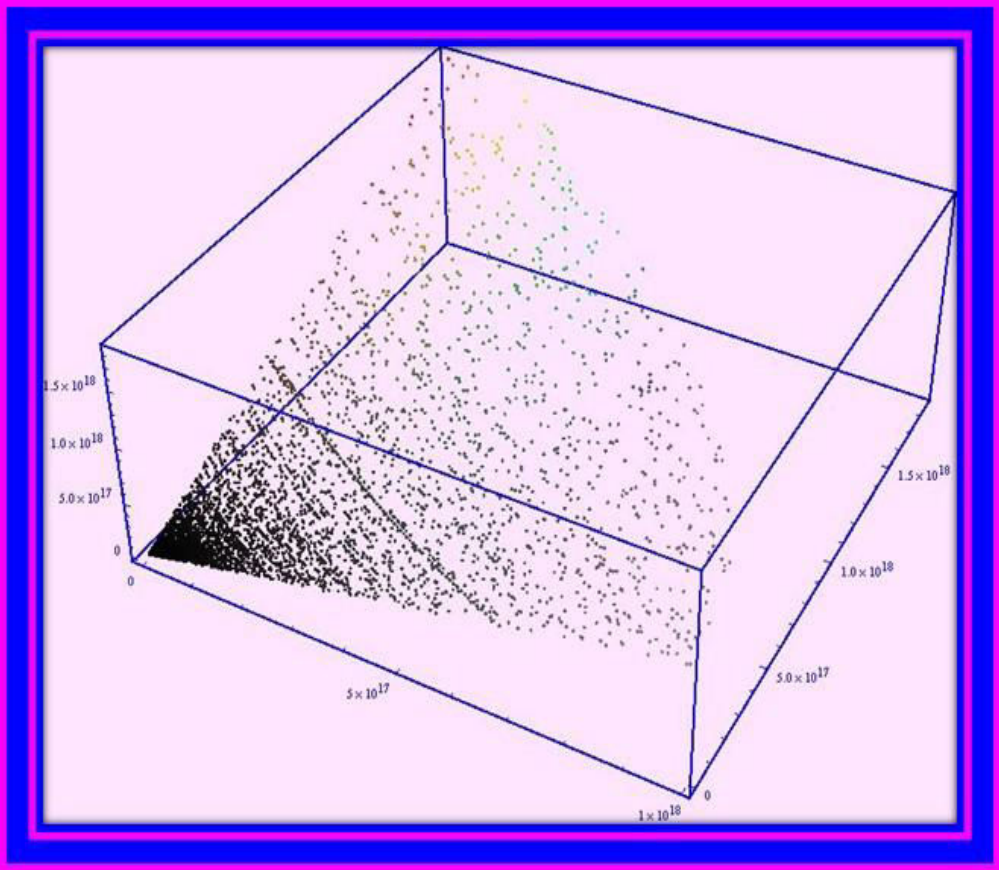


Figure 4a.3

4.4 OBSRVATIONS

For all Pythagorean Triangles (X, Y, Z), with

$X + Y + Z = m^2 + n^2 + \beta^5$ and $0 < \beta < 100$, we observe that,

1. Except for $\beta = 49$ and $\beta = 98$, for each value of β, there are three pairs of (m, n) which will give three Pythagorean Triangles (X, Y, Z).
2. For $\beta = 49$ and $\beta = 98$, there are five Pythagorean Triangles correspondingly.
3. $X + Y + Z = 0 (\text{mod } 2)$.

CHAPTER 4(B)

SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS A HEX IC*

4.5 INTRODUCTION

In the order of powers, after quintic comes hexic. Gopalan and Vijayasankar [35] have determined the solutions of hexic Diophantine equation. In this chapter, an effort has been made to find Special Pythagorean Triangles with perimeter as sum of two squares and a hexic. Such triangles with two sides consecutive are also investigated. Various 3D graphs are also plotted.

4.6 METHOD OF ANALYSIS

We have seen in the previous chapters, the primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2, \qquad \textbf{(4.6.1)}$$

are given by Niven and Zukerman [60], where

$$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2, \qquad \textbf{(4.6.2)}$$

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

***Sum of two legs is a hexic:** In that case,

$$X + Y = \beta^6, \quad \beta \in N, \qquad \textbf{(4.6.3)}$$

$$\text{i.e., } m^2 + 2mn - n^2 - \beta^6 = 0. \qquad \textbf{(4.6.4)}$$

The solutions of equations (4.6.4) are given, by theory of equations, as

$$\beta = -(m^2 + 2mn - n^2)^{1/6},$$

$$\beta = (m^2 + 2mn - n^2)^{1/6},$$

$$\beta = -(-1)^{1/3}(m^2 + 2mn - n^2)^{1/6},$$

$$\beta = (-1)^{1/3}(m^2 + 2mn - n^2)^{1/6},$$

$$\beta = -(-1)^{2/3}(m^2 + 2mn - n^2)^{1/6},$$

$$\beta = (-1)^{2/3}(m^2 + 2mn - n^2)^{1/6}. \qquad \textbf{(4.6.5)}$$

Only the positive integral values of β are taken as the solutions.

***Perimeter is sum of two squares and a hexic:** If $X + Y = \beta^6$ then from equation (4.6.2), we have

$$X + Y + Z = \beta^6 + m^2 + n^2. \qquad \textbf{(4.6.6)}$$

Equations (4.6.2) and (4.6.4) generate (X, Y, Z) which satisfy equations (4.6.1) and (4.6.3) in correspondence with (4.6.6).

Solving the equation (4.6.4), using software Mathematica,

$m^2 + 2mn - n^2 - \beta^6 = 0$, for $n < m$; $0 < n < 10^9$; $0 < m < 10^9$; $0 < \beta < 10^9$; we get the 2685 values for (X, Y, Z). Out of these 2685 values for (X, Y, Z), only 136 are primitives. Following are the few examples in Table 4b.1 (precisely 65) with $0 < \beta < 10^2$:-

*This Paper is published in World Wide Journal of Multidisciplinary Research and Development WWJMRD 23; 9(06): 20-25

Table 4b.1: Special Pythagorean Triangles (X, Y, Z)

S.N.	m	N	β	$X = m^2 - n^2$	$Y = 2mn$	$Z = m^2 + n^2$
1	245	196	7	21609	96040	98441
2	265	114	7	57229	60420	83221
3	287	70	7	77469	40180	87269
4	3485	3094	17	2572389	21565180	21718061
5	3757	1734	17	11108293	13029276	17121805
6	4033	1136	17	14974593	9162976	17555585
7	6615	5292	21	15752961	70013160	71763489
8	7155	3078	21	41719941	44046180	60668109
9	7749	1890	21	56474901	29291220	63619101
10	8947	5474	23	50084133	97951756	110013485
11	8993	5290	23	52889949	95145940	108858149
12	12013	156	23	144287833	3748056	144336505
13	21359	16368	31	188295457	699208224	724118305
14	21449	15738	31	212374957	675128724	707744245
15	24025	7688	31	518095281	369408400	636305969
16	30625	24500	35	337640625	1500625000	1538140625
17	33125	14250	35	894203125	944062500	1300328125
18	35875	8750	35	1210453125	627812500	1363578125
19	48749	47068	41	161068377	4589035864	4591861625
20	48865	43818	41	467771101	4282333140	4307805349
21	66625	2378	41	4433235741	316868500	4444545509
22	73555	67116	47	905780569	9873434760	9914895481
23	77897	41064	47	4381690513	6397524816	7754194705
24	81733	30926	47	5723865813	5055349516	7636700765
25	83221	80030	49	520933941	13320353260	13330535741
26	84035	67228	49	2542277241	11299009960	11581485209
27	87269	49980	49	5117877961	8723409240	10113878761
28	90895	39102	49	6732934621	7108352580	9790867429
29	98441	24010	49	9114150381	4727136820	10267110581
30	105175	14168	49	10861048401	2980238800	11262512849
31	94095	83538	51	1875271581	15721016220	15832466469
32	101439	46818	51	8097945597	9498342204	12481795845
33	108891	30672	51	10916478297	6679809504	12798021465
34	178605	142884	63	11483908569	51039593640	52315583481
35	193185	83106	63	30413836989	32109665220	44227051461
36	209223	51030	63	41170202829	21353299380	46378324629
37	241569	147798	69	36511332957	71406830124	80199830565
38	242811	142830	69	38556772821	69361390260	79357590621
39	324351	4212	69	105185830257	2732332824	105221312145
40	256789	195300	71	27798500521	100301783400	104082680521
41	274415	124392	71	59830222561	68270061360	90776961889
42	307501	60492	71	90897582937	37202700984	98216147065
43	282437	191844	73	42966538633	108367687656	116574779305
44	299665	131546	73	72494762109	78839464180	107103462341
45	343705	52274	73	115400555949	35933670340	120865698101
46	326095	260876	77	38281661649	170140718440	174394236401
47	352715	151734	77	101384664469	107037715620	147431077981

48	381997	93170	77	137241059109	71181320980	154602356909
49	359371	236052	79	73426968937	169660486584	184868062345
50	405665	112338	79	151944265981	91143189540	177183918469
51	435625	386750	85	40193578125	336955937500	339344703125
52	469625	216750	85	173567078125	203582437500	267528203125
53	504125	142000	85	233978015625	143171500000	274306015625
54	514865	332682	89	154408655101	342572635860	375763281349
55	533021	266110	89	213296854341	283684436620	354925918541
56	652405	57188	89	422361816681	74619474280	428902751369
57	538265	430612	91	104302515681	463566736360	475155904769
58	582205	250458	91	276233452261	291635799780	401691871789
59	630539	153790	91	373928066421	193941185620	421230794621
60	576693	441936	93	137267388153	509722795296	527882244345
61	579123	424926	93	154821343653	492168839796	515945554605
62	648675	207576	93	377691459849	269298723600	463867051401
63	669593	417130	97	274357348749	558614656180	622352222549
64	719837	268884	97	445866701113	387105303816	590463912025
65	799765	131726	97	622272316149	210699688780	656975794301

The following Table 4b.2 verifies that these are Pythagorean Triangles for first seven (X, Y, Z) in Table 4b.1while Table 4b.3 verifies that these Pythagorean Triangles (X, Y, Z) in Table 4b.2 have perimeters as sum of two squares and a hexic.

Table 2b.2: Verification of $X^2 + Y^2 = Z^2$

S.N.	$\mathbf{X^2}$	$\mathbf{Y^2}$	$\mathbf{X^2 + Y^2}$	$\mathbf{Z^2}$
1	466948881	9223681600	9690630481	9690630481
2	3275158441	3650576400	6925734841	6925734841
3	6001445961	1614432400	7615878361	7615878361
4	6617185167321	465056988432400	471674173599721	471674173599721
5	123394173373849	169762033084176	293156206458025	293156206458025
6	224238435515649	83960129176576	308198564692225	308198564692225
7	248155780267521	4901842573185600	5149998353453121	5149998353453121

Table 2b.3: $X + Y = \beta^6$ and $X + Y + Z = \beta^6 + m^2 + n^2$

S.N.	$\mathbf{X + Y = \beta^6}$	$\mathbf{X + Y + Z = \beta^6 + m^2 + n^2}$
1	$117649 = 7^6$	$216090 = 7^6 + 245^2 + 196^2$
2	$117649 = 7^6$	$200870 = 7^6 + 265^2 + 114^2$
3	$117649 = 7^6$	$204918 = 7^6 + 287^2 + 70^2$
4	$24137569 = 17^6$	$45855630 = 17^6 + 3485^2 + 3094^2$
5	$24137569 = 17^6$	$41259374 = 17^6 + 3757^2 + 1734^2$
6	$24137569 = 17^6$	$41693154 = 17^6 + 4033^2 + 1136^2$
7	$85766121 = 21^6$	$157529610 = 21^6 + 6615^2 + 5292^2$

***One Leg and Hypotenuse are Consecutive:**

We know, from previous chapters, that in such cases,

$$m = n + 1. \qquad \textbf{(4.6.7)}$$

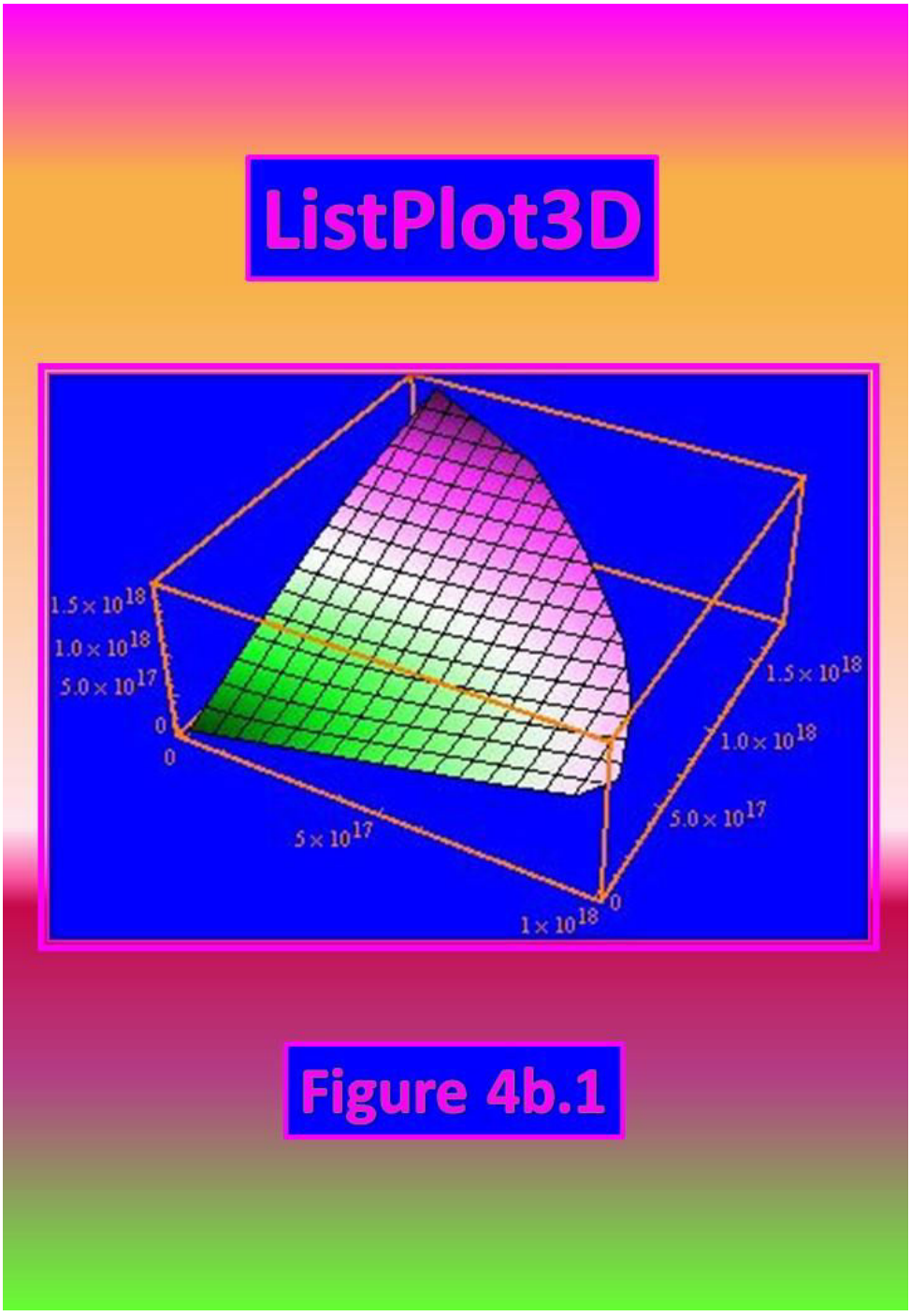
Substituting the value of m from equation (4.6.7) in equations (4.6.2) and (4.6.4), we get the Diophantine equation,

$$2n^2 + 4n + 1 - \beta^6 = 0. \qquad \textbf{(4.6.8)}$$

Using the software Mathematica, equation (4.6.8) was solved for $0 < n < 10^{13}$ and $0 < \beta < 10^5$ and it was found that it has no solution. It needs further investigations to find whether any solution of (4.6.8) exists for $n \geq 10^{13}$.

4.7 3D PLOTS

For $n < m$, $0 < n < 10^9$, $0 < m < 10^9$, $0 < \beta < 10^9$, we get the 2685 values for (X, Y, Z) by solving equation (4.6.4) as obtained in section 4.6. Plotting these values as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D, we get figure 4b.1, Figure 4b.2 and Figure 4b.3 respectively. They are as follows:



ISBN: 978-93-93810-76-2

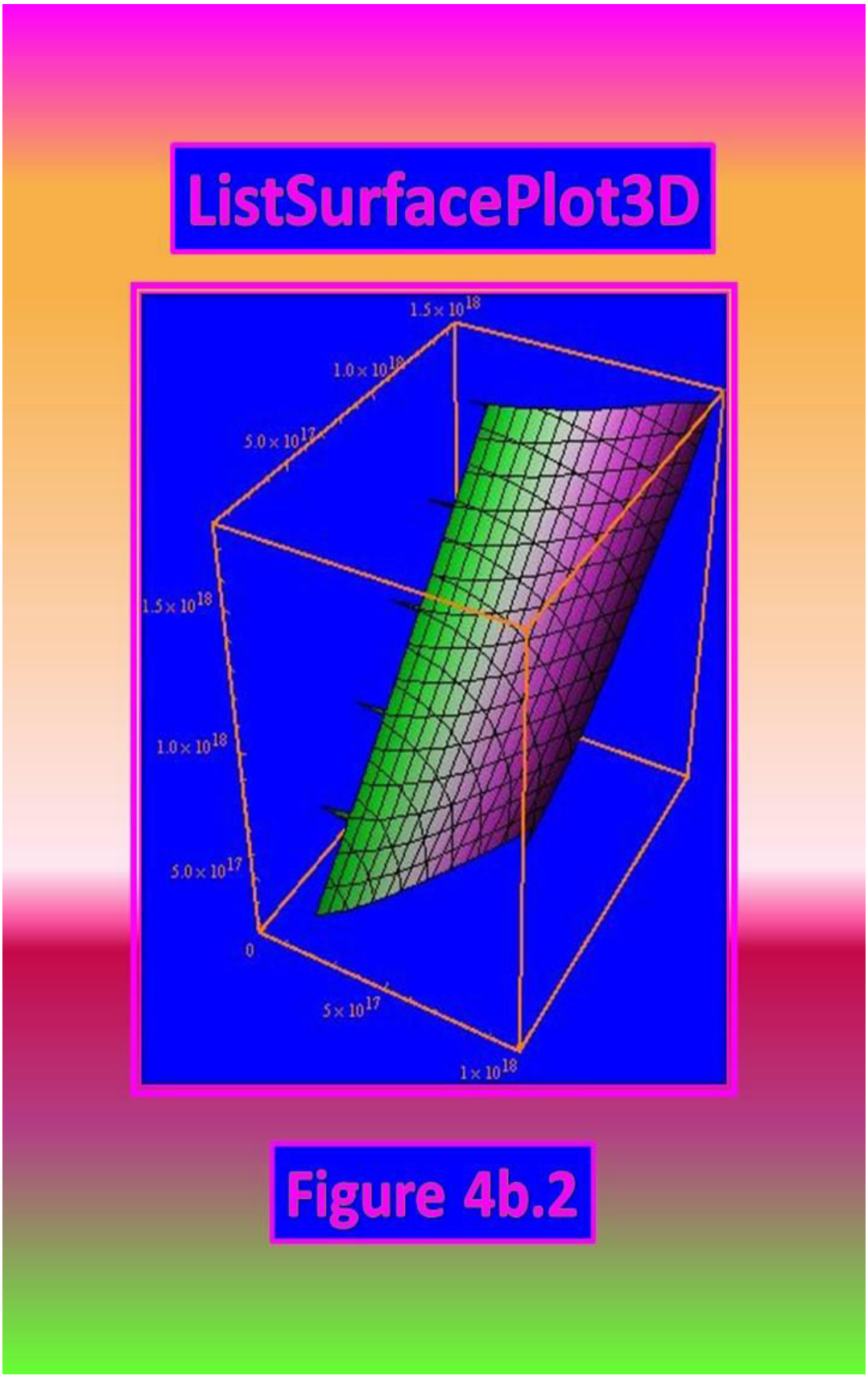


Figure 4b.2

ISBN: 978-93-93810-76-2

ListPointPlot3D

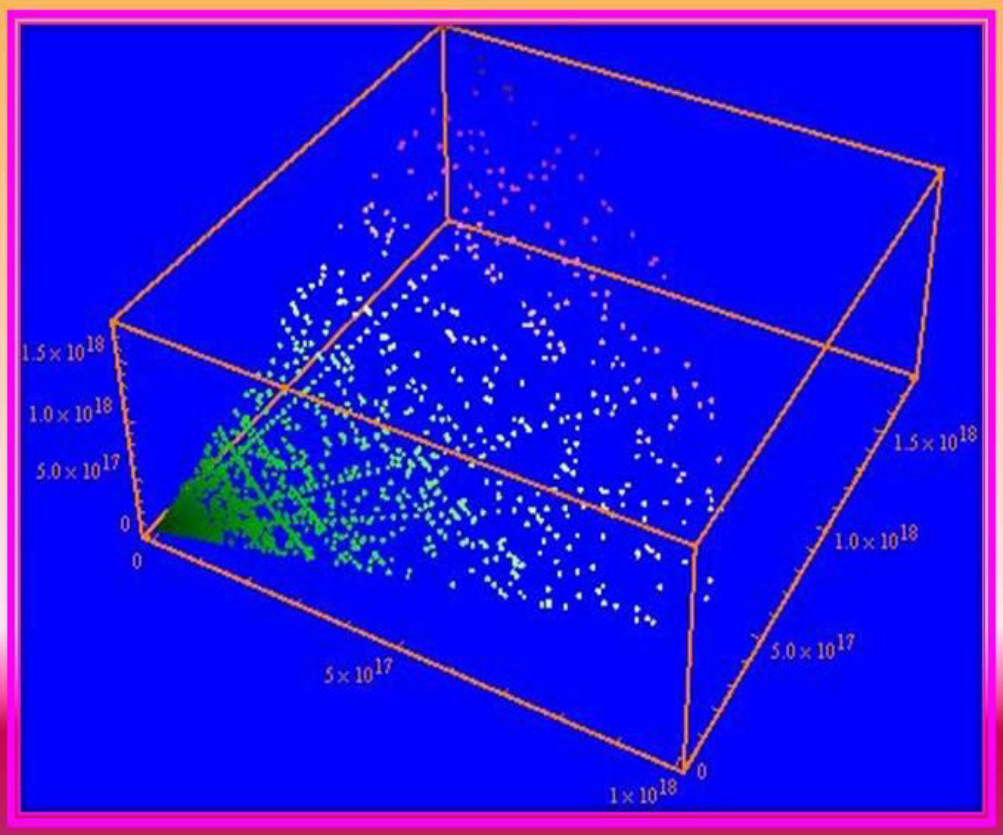


Figure 4b.3

 ISBN: 978-93-93810-76-2

4.8 OBSERVATIONS

The following observations are made for $\beta < 100$:-

1. Except for $\beta = 49$ and $\beta = 98$, there are three Pythagorean Triangles corresponding to each β.
2. For $\beta = 49$ and $\beta = 98$, there are six Pythagorean Triangles satisfying $X + Y = \beta^6$.
3. $Y + Z - X = 0 \pmod 2$.

ISBN: 978-93-93810-76-2

CHAPTER FIVE

SPECIAL PYTHAGOREAN TRIANGLES IN RELATION WITH HEPTIC AND OCTIC DIOPHANTINE EQUATIONS

CHAPTER 5(A)

SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS A HEPTIC*

5.1 INTRODUCTION

After obtaining the special Pythagorean triangles, each of whose perimeters is the sum of two distinct squares and hexic in the previous chapter, made us explore the path of discovery of Pythagorean Triangles (X, Y, Z) with sum of their legs a heptic. The existence of such triangles with two sides consecutive is also investigated. We also attempt to draw the graphs of these triangles as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D.

5.2 METHOD OF ANALYSIS

Niven and Zuckerman [60], as we have seen in previous chapters, have given the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$, **(5.2.1)**

as, $X = m^2 - n^2$, $Y = 2mn$, $Z = m^2 + n^2$, **(5.2.2)**

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

*** Sum of two legs is a heptic**: When the sum of two legs X and Y of Pythagorean Triangles is a heptic then we get the following equation: -

$X + Y = \beta^7$, where $\beta \in N$. **(5.2.3)**

Or, by (5.2.2), $m^2 + 2mn - n^2 - \beta^7 = 0$. **(5.2.4)**

By theory of equations, the solutions of equation (5.2.4) and subsequently equation (5.2.3) are given by,

$\beta = (m^2 + 2mn - n^2)^{1/7}$,

$\beta = -(-1)^{1/7} (m^2 + 2mn - n^2)^{1/7}$,

$\beta = (-1)^{2/7} (m^2 + 2mn - n^2)^{1/7}$,

$\beta = -(-1)^{3/7} (m^2 + 2mn - n^2)^{1/7}$,

$\beta = (-1)^{4/7} (m^2 + 2mn - n^2)^{1/7}$,

$\beta = -(-1)^{5/7} (m^2 + 2mn - n^2)^{1/7}$,

$\beta = (-1)^{6/7} (m^2 + 2mn - n^2)^{1/7}$. **(5.2.5)**

From these values, only those values are taken for which β is a positive integer.

*** Perimeter is sum of two squares and a heptic:** We see that if the sum of two legs is a heptic and hypotenuse is sum of two squares then the perimeter is given by equations (5.2.2) and (5.2.3),

$X + Y + Z = \beta^7 + m^2 + n^2$. **(5.2.6)**

Equations (5.2.2) & (5.2.4) give Pythagorean triangles (X, Y, Z)

which satisfy equation (5.2.6). Now, solving the Diophantine equation (5.2.4) for $n < m$, $0 < n < 10^{10}$, $0 < m < 10^{10}$, $0 < \beta < 10^{10}$, we get the 1435 values for (X, Y, Z) out of which only 55 are primitives! Following are few examples in Table 5a.1: -

* This Paper is published in Global Journal of Computational Science and Mathematics (GJCSM); 1(2) (2011); 81-86.

Table 5a.1: Special Pythagorean Triangles (X, Y, Z)

S.N.	m	n	β	$X = m^2 - n^2$	$Y = 2mn$	$Z = m^2 + n^2$
1	644	567	7	93247	730296	736225
2	686	343	7	352947	470596	588245
3	784	147	7	593047	230496	636265
4	872	37	7	759015	64528	761753
5	14371	12724	17	44625465	365713208	368425817
6	14739	9826	17	120687845	289650828	313788397
7	16643	4658	17	255292485	155046188	298686413
8	18207	2312	17	326149505	84189168	336840193
9	41262	40733	23	43375355	3361450092	3361729933
10	48208	12949	23	2156334663	1248490784	2491687865
11	48668	12167	23	2220538335	1184287112	2516610113
12	49144	11389	23	2285423415	1119402032	2544842057
13	119164	89373	31	6212525767	21300088344	22187592025
14	133010	44293	31	15729790251	11782823860	19653529949
15	134540	41323	31	16393421271	11119192840	19808601929
16	164486	1395	31	27053698171	458915940	27057590221
17	331961	171830	41	80672556621	114081717260	139723654421
18	337881	154652	41	90246329057	104507944824	138080811265
19	344605	137842	41	99752189061	95002084820	137753022989
20	352149	121360	41	109280668601	85473605280	138737167801
21	508446	406409	47	93349059635	413274060828	423685610197
22	521324	329141	47	163444915095	343178205368	380112510857
23	582112	168487	47	310466511375	196156609088	367242249713
24	622938	103823	47	377272536515	129350583948	398830967173
25	582547	560210	49	25525763109	652697309740	653196251309
26	588245	470596	49	124571584809	553651488040	567492775241
27	610883	349860	49	250776020089	427447052760	495580059289
28	636265	273714	49	329913796429	348309276420	479752504021
29	689087	168070	49	446593368669	231629704180	503088418469
30	736225	99176	49	532191371649	146031701200	551863129601
31	761753	67266	49	575742918253	102480154596	584792347765
32	1408428	1240029	63	445997510343	3492983128824	3521341352025
33	1500282	750141	63	1688134559643	2250846079524	2813557599405
34	1714608	321489	63	2836525416543	1102455222624	3043235770785
35	1907064	80919	63	3630345215535	308635423632	3643440984657
36	2147466	1789555	71	1409103123131	7686017035260	7814117319181
37	2268450	1174553	71	3766290652691	5328829505700	6525440152309
38	2517234	625723	71	5944937738027	3150182420364	6727996283485
39	2907834	112145	71	8442922070531	652198087860	8468075072581
40	2360747	2046336	73	1385635373113	9661763145984	9760617422905
41	2510483	1262424	73	4708810537513	6338587981584	7896239249065
42	2723119	778034	73	6810040183005	4237358336092	8020713993317
43	3033223	321492	73	9097084661665	1950313857432	9303798873793
44	3111020	2719891	79	2280638388519	16923270597640	17076252492281
45	3179422	2172675	79	5388207598459	13815701387700	14829240909709
46	3357658	1529045	79	8935888632939	10268020353220	13611845856989
47	3944312	493039	79	15314509697823	3889399288336	15800684608865

48	4795587	3467440	89	10974514520969	33256820374560	35020794828169
49	4934783	2819876	89	16400382601713	27830952293816	32303783912465
50	5138715	2209304	89	21525367686809	22705967208720	31287416015641
51	6392247	269314	89	40788291678413	3443043217116	40933351739605
52	6388711	5476038	97	10828636064077	69969648414036	70802620418965
53	6652163	3876508	97	29223958304505	51574326173608	59278586852633
54	7189931	2436834	97	45756947841205	35041336636908	57633267728317
55	8024191	1097648	97	63182810072577	17615474405536	65592472336385

Table 5a.2 gives verification of $X^2 + Y^2 = Z^2$ for first ten Pythagorean Triangles (X, Y, Z), Table 5a.3 gives verification of $X + Y = \beta^7$ and $X + Y + Z = \beta^7 + m^2 + n^2$.

Table 5a.2: Verification of $X^2 + Y^2 = Z^2$

S.N	**X^2**	**Y^2**	**$X^2 + Y^2$**	**Z^2**
1	8695003009	533332247616	542027250625	542027250625
2	124571584809	221460595216	346032180025	346032180025
3	351704744209	53128406016	404833150225	404833150225
4	576103770225	4163862784	580267633009	580267633009
5	1991432126466225	1337461505056 51264	135737582632 117489	135737582632 117489
6	1456555593074 4025	8389760216108 5584	984631580918 29609	984631580918 29609

Table 5a.3: Verification of $X + Y = \beta^7$; $X + Y + Z = \beta^7 + m^2 + n^2$

S.N.	**$X + Y = \beta^7$**	**$X + Y + Z = \beta^7 + m^2 + n^2$**
1	$823543 = 7^7$	$1559768 = 7^7 + 644^2 + 567^2$
2	$823543 = 7^7$	$1411788 = 7^7 + 686^2 + 343^2$
3	$823543 = 7^7$	$1459808 = 7^7 + 784^2 + 147^2$
4	$823543 = 7^7$	$1585296 = 7^7 + 872^2 + 37^2$
5	$410338673 = 17^7$	$778764490 = 17^7 + 14371^2 + 12724^2$
6	$410338673 = 17^7$	$724127070 = 17^7 + 14739^2 + 9826^2$

*** One Leg and Hypotenuse are Consecutive:**

If one leg and a hypotenuse are consecutive then in such cases, we know that $m = n + 1$. Substituting this value of m in equations (5.2.2) and (5.2.4), we get the Diophantine equation,

$$2n^2 + 4n + 1 - \beta^7 = 0. \qquad \textbf{(5.2.7)}$$

Using the software Mathematica, equation (5.2.7) was solved for $0 < n < 10^{13}$; $0 < \beta < 10^{13}$ and it was found that it has no solution. It needs further investigations to find whether the solution of equation (5.2.7) exists for $n \geq 10^{13}$.

5.3 3D PLOTS

For 1435 values for (X, Y, Z), which are obtained in section 5.2, following graphs are plotted as given in Figure 5a.1, 5a.2 and 5a.3: -

ISBN: 978-93-93810-76-2

ListPlot3D

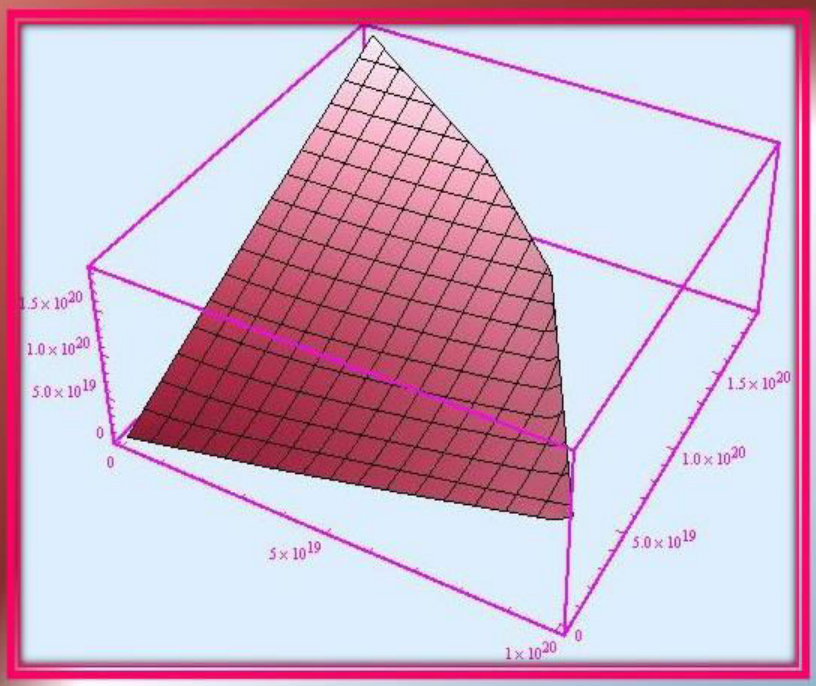
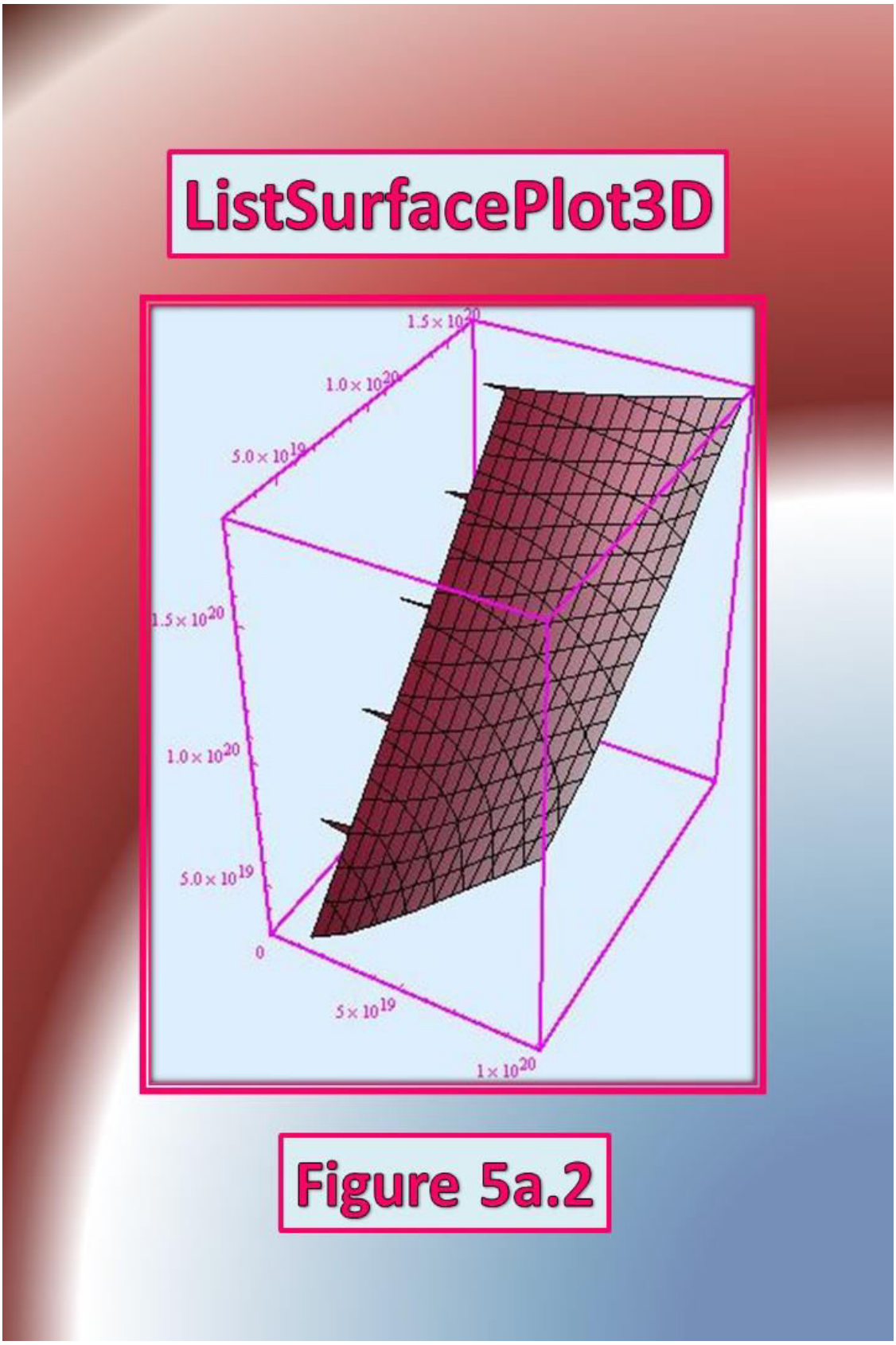


Figure 5a.1

ISBN: 978-93-93810-76-2



ISBN: 978-93-93810-76-2

ListPointPlot3D

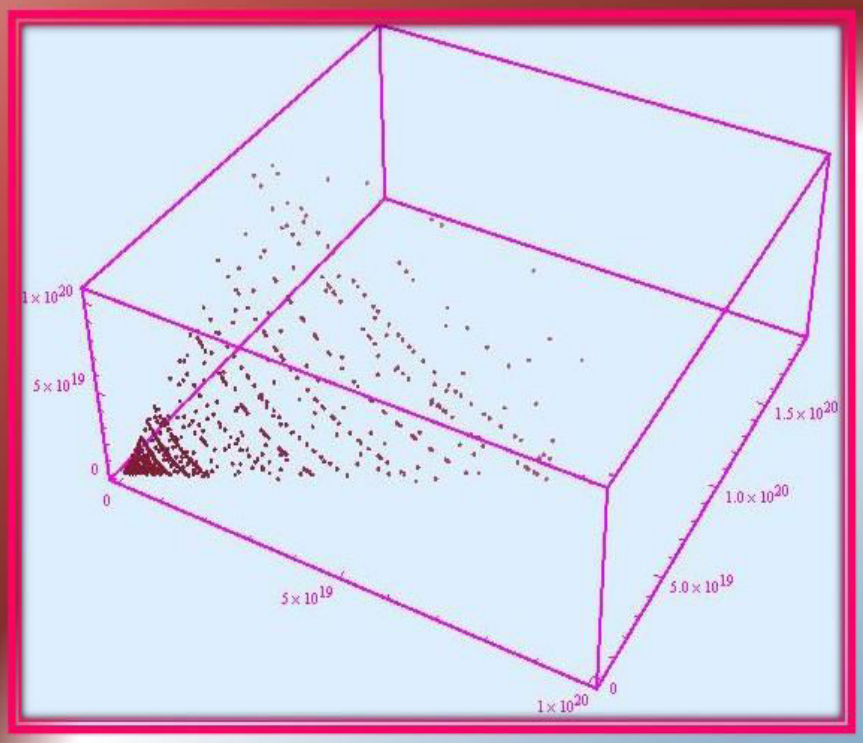


Figure 5a.3

 ISBN: 978-93-93810-76-2

5.4 OBSERVATIONS

We also observe that,

1. $X + Y + Z = 0 (\text{mod } 2)$.
2. $(Y + Z – X)^2 = 2(Y + Z)(Z – X)$.
3. For $\beta < 100$, except for $\beta = 49$ and $\beta = 98$, there are four Pythagorean Triangles (X, Y, Z) for each β.
4. For $\beta = 49$ and $\beta = 98$, seven Pythagorean Triangles

(X, Y, Z) satisfy $X + Y = \beta^7$.

CHAPTER 5(B)

SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS AN OCTIC*

5.5 INTRODUCTION

In this chapter, we strive to extend the result obtained in the previous chapter. We explore the possibility of existence of any Pythagorean Triangle with sum of its legs as an octic. The existence of such triangles with two sides consecutive is also investigated. We also make an attempt to plot these special Pythagorean Triangles (X, Y, Z).

5.6 METHOD OF ANALYSIS

As we have seen in the previous chapters, the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$, **(5.6.1)**

is given by Niven and Zuckerman [60]

$X = m^2 - n^2$, $Y = 2mn$, $Z = m^2 + n^2$, **(5.6.2)**

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

*** Sum of two legs is an octic:** We consider the case when

$X + Y = \beta^8$, where $\beta \in N$. **(5.6.3)**

i.e., $m^2 + 2mn - n^2 - \beta^8 = 0$ **(5.6.4)**

Then, by the theory of equation, the solutions of equation (5.6.4)

are given by,

$\beta = -(m^2 + 2mn - n^2)^{1/8}$,

$\beta = (m^2 + 2mn - n^2)^{1/8}$,

$\beta = -i(m^2 + 2mn - n^2)^{1/8}$,

$\beta = i(m^2 + 2mn - n^2)^{1/8}$,

$\beta = -(-1)^{1/4}(m^2 + 2mn - n^2)^{1/8}$,

$\beta = (-1)^{1/4}(m^2 + 2mn - n^2)^{1/8}$,

$\beta = -(-1)^{3/4}(m^2 + 2mn - n^2)^{1/8}$,

$\beta = (-1)^{3/4}(m^2 + 2mn - n^2)^{1/8}$. **(5.6.5)**

From these values only those values are taken for which β is a positive integer.

*** Perimeter is sum of two squares and an octic:** If the sum of two legs is an octic and the hypotenuse being sum of two squares, the equations (5.6.2) and (5.6.3) give

$X + Y + Z = \beta^8 + m^2 + n^2$ **(5.6.6)**

Equations (5.6.2) and (5.6.4) will generate X, Y and Z, which will satisfy the octic Diophantine equation (5.6.6). Solving the Diophantine equation (5.6.4), using software Mathematica,

$m^2 + 2mn - n^2 - \beta^8 = 0$ **(5.6.7)**

*This Paper has been published in Global Journal of Applied Mathematics and Mathematical Sciences (GJ-AMMS); 4(1); (January-June 2011); 37-44.

for $n < m$, $0 < n < 10^{10}$, $0 < m < 10^{10}$, $0 < \beta < 10^{10}$, we get the 842 values for the Pythagorean triangles (X, Y, Z). Out of these 842 values only 42 Pythagorean Triangles are primitives.

Following Table 5b.1 gives the first seven examples of these Pythagorean Triangles (X, Y, Z) when $0 < \beta < 50$. Table 5b.2 verifies that $X^2 + Y^2 = Z^2$ while Table 5b.3 verifies that these Primitive Pythagorean Triangles have perimeters as sum of two distinct squares and an octic.

Table 5b.1: Primitive Pythagorean Triangles with $\beta < 50$

S.N.	M	n	β	$X = m^2 - n^2$	$Y = 2mn$	$Z = m^2 + n^2$
1	1781	1020	7	2131561	3633240	4212361
2	75149	9430	17	5558447301	1417310140	5736297101
3	207965	117478	23	29448360741	48862624540	57050521709
4	753505	221858	31	518548812861	334342224580	616990757189
5	2643989	195228	41	6952563860137	1032361368984	7028791804105
6	4488409	428810	47	19961937335181	3849349326580	20329693367381
7	4212361	2710682	49	10396188289197	22836742280404	25091782099445

Table 5b.2: $X^2 + Y^2 = Z^2$

S.N.	X^2	Y^2	$X^2 + Y^2 = Z^2$
1	4543552296721	13200432897600	17743985194321
2	30896336397994184601	2008768032946819600	32905104430941004201
3	867205950332070069081	2387556076937010211600	3254762027269080280681
4	268892871319552399005321	11178472371371031561764 00	3806775944566555551817 21
5	4833814422928310209765 8769	1065769996170518597192 256	4940391422545362069485 1025
6	398478942173693123540 302761	148174902380418994944 96400	413296432411735023034 799161
7	1080807309444368457069 04809	5215167979815916861624 03216	6295975289260285318693 08025

Table 5b.3: $X + Y = \beta^8$, $X + Y + Z = \beta^8 + m^2 + n^2$

S.N.	X + Y	β^8	X + Y + Z	$\beta^8 + m^2 + n^2$
1	5764801	$5764801 = 7^8$	9977162	9977162
2	6975757441	$6975757441 = 17^8$	12712054542	12712054542
3	78310985281	$78310985281 = 23^8$	135361506990	135361506990
4	852891037441	$852891037441 = 31^8$	1469881794630	1469881794630
5	7984925229121	$7984925229121 = 41^8$	15013717033226	15013717033226
6	23811286661761	$23811286661761 = 47^8$	44140980029142	44140980029142
7	33232930569601	$33232930569601 = 49^8$	58324712669046	58324712669046

*** One leg and Hypotenuse are Consecutive**

In such cases, as discussed in the previous chapters, $m = n + 1$.

Substituting the value of m in equations (5.6.2) and (5.6.4), we get the Diophantine equation,

$$2n^2 + 4n + 1 - \beta^8 = 0. \qquad \textbf{(5.6.7)}$$

An attempt was made to solve the equation (5.6.7) using the software Mathematica for $0 < n < 10^{13}$ and $0 < \beta < 10^{13}$ and it was found that it had no solution. It needs further investigations to find whether any solution of equation (5.6.7) exists for $n \geq 10^{13}$.

5.7 3D PLOTS

For $n < m$, $0 < n < 10^{10}$, $0 < m < 10^{10}$, $0 < \beta < 10^{10}$, we get the 842 values for (X, Y, Z). Plotting these values as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D we get the following graphs:-

 ISBN: 978-93-93810-76-2

ListPlot3D

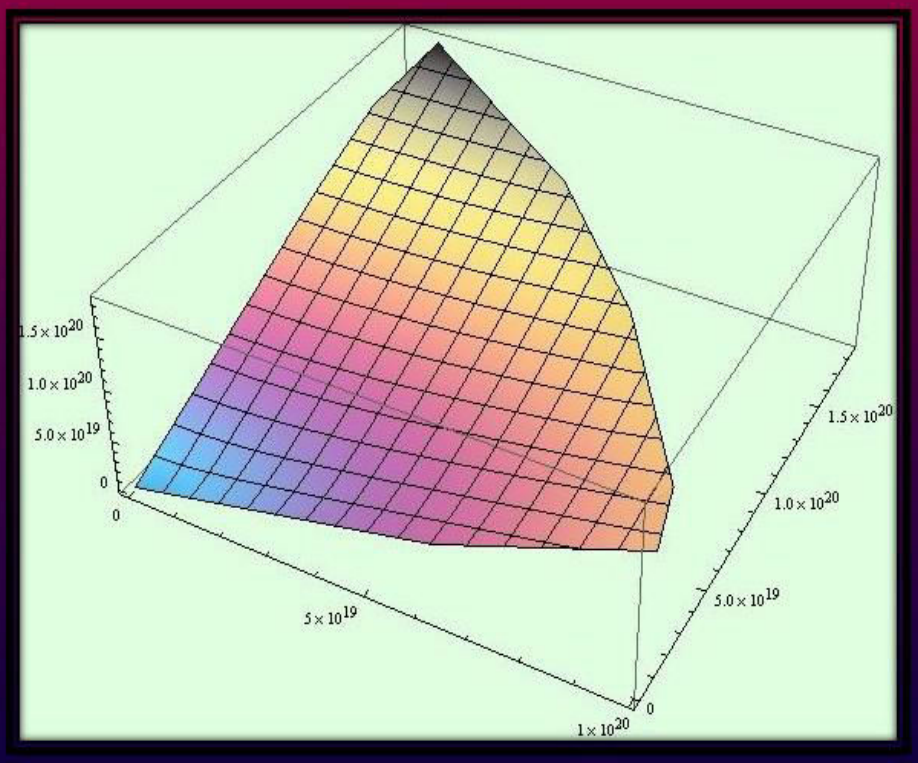


Figure 5b.1

ISBN: 978-93-93810-76-2

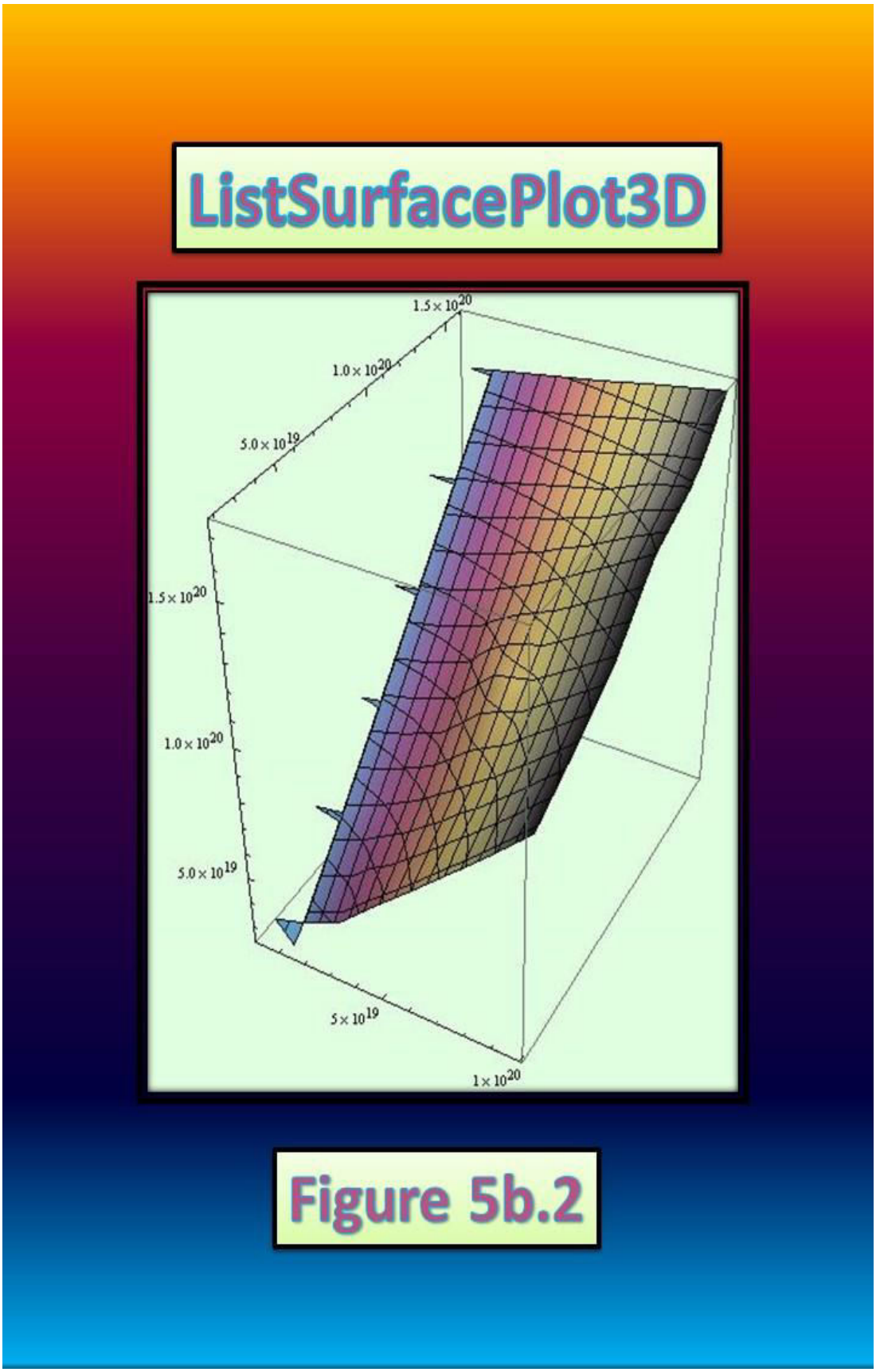


Figure 5b.2

ISBN: 978-93-93810-76-2

ListPointPlot3D

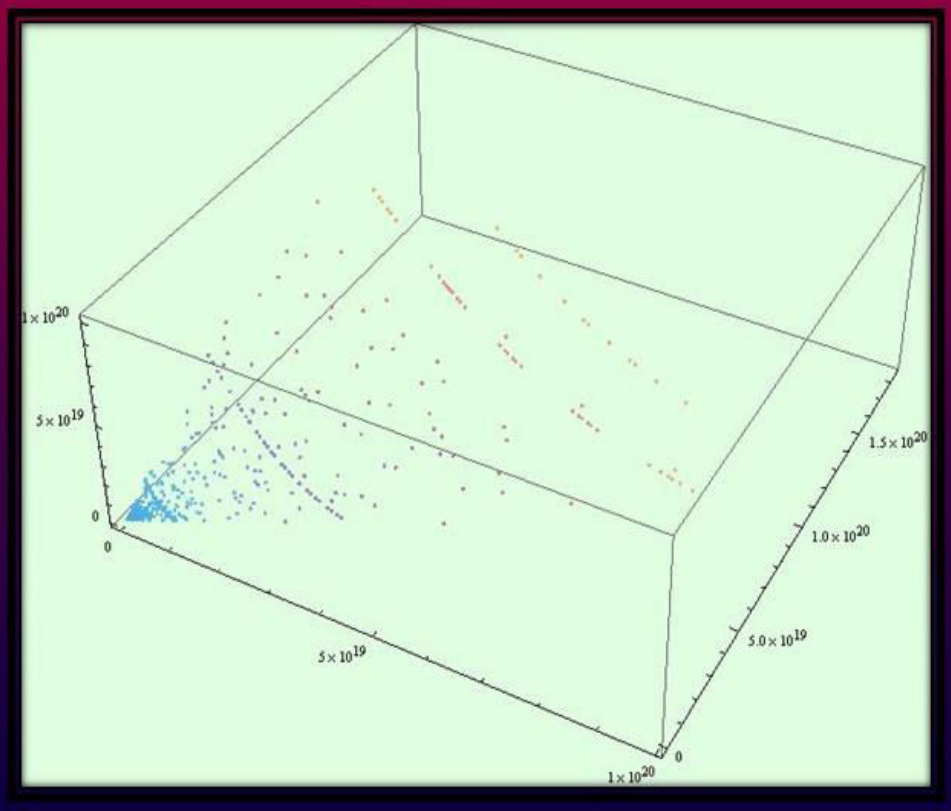


Figure 5b.3

 ISBN: 978-93-93810-76-2

5.8 OBSERVATIONS

Also, we observe that,

1. For $\beta < 100$, except for $\beta = 49$ and $\beta = 98$, there are four Pythagorean Triangles (X, Y, Z) for each β.
2. For $\beta = 49$ and $\beta = 98$, eight Pythagorean Triangles (X, Y, Z) satisfy $X + Y = \beta^8$.
3. $X + Y + Z = 0 \pmod 2$.

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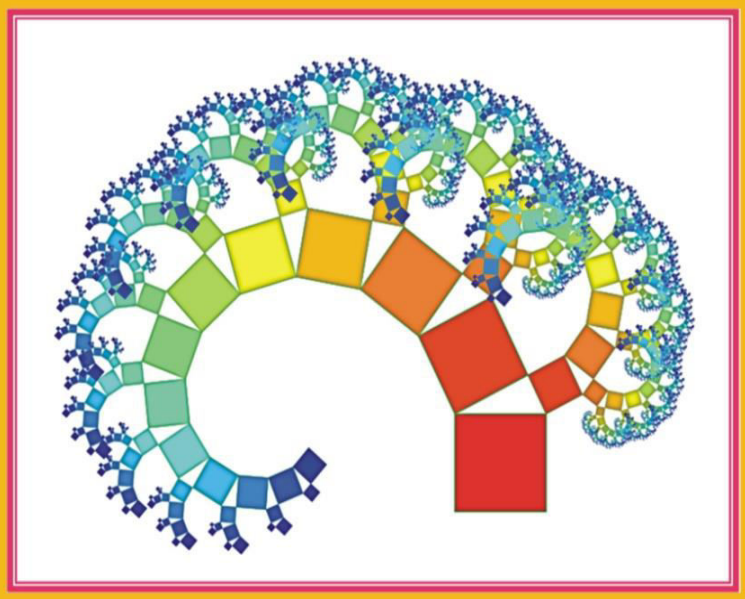
CHAPTER SIX



SPECIAL PYTHAGOREAN TRIANGLES IN TERMS OF NONIC AND DECIC DIOPHANTINE EQUATIONS

CHAPTER 6(A)

SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS A NONIC*

6.1 INTRODUCTION

It is a natural question whether the results obtained in the previous chapter can be extended further. That inspired us to search for Pythagorean Triangle, with sum of its legs a nonic. The existence of such triangles with two consecutive sides is also looked into. An attempt has been made to plot such Special Pythagorean Triangles (X, Y, Z) with perimeter as sum of two squares and a nonic.

6.2 METHOD OF ANALYSIS

Niven and Zuckerman [60] gave primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$, **(6.2.1)**

as, $X = m^2 - n^2$, $Y = 2mn$, $Z = m^2 + n^2$, **(6.2.2)**

for some integers m, n of opposite parity such that $m > n > 0$ and

$(m, n) = 1$.

***Sum of two legs is a nonic:** If X and Y are two legs of a Pythagorean Triangle and if for some $\beta \in N$,

$X + Y = \beta^9$, **(6.2.3)**

then equation (6.2.2) gives

$m^2 + 2mn - n^2 - \beta^9 = 0$. **(6.2.4)**

The solutions of the nonic Diophantine equation (6.2.4) are given by,

$\beta = (m^2 + 2\,m\,n - n^2)^{1/9}$,

$\beta = -(-1)^{1/9} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = (-1)^{2/9} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = -(-1)^{1/3} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = (-1)^{4/9} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = -(-1)^{5/9} (m^2 + 2\,mn - n^2)^{1/9}$,

$\beta = (-1)^{2/3} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = -(-1)^{7/9} (m^2 + 2mn - n^2)^{1/9}$,

$\beta = (-1)^{8/9} (m^2 + 2mn - n^2)^{1/9}$. **(6.2.5)**

From these, only positive integral values of β are sought for.

***Perimeter is sum of two squares and a nonic:** When the sum of two legs is a nonic, then its perimeter as the sum of two squares and a nonic is obtained from equations (6.2.2) and (6.2.3),

$X + Y + Z = \beta^9 + m^2 + n^2$. **(6.2.6)**

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* This Paper is published in International Journal: Mathematical Modelling and Applied Computing (MMAC); 2 (1) (2011); 37-42

We see that, equations (6.2.2) and (6.2.4) generate Pythagorean triangles (X, Y, Z) with perimeters as $\beta^9 + m^2 + n^2$.

Solving the Diophantine equation, using software Mathematica,

$m^2 + 2\,m\,n - n^2 - \beta^9 = 0$, for $n < m$, $0 < n < 10^{12}$, $0 < m < 10^{12}$,

$0 < \beta < 10^7$, we get the 1100 values for (X, Y, Z) which will consume many pages to write down! Out of these 1100 values, only 61 are Primitive Pythagorean Triangles with above mentioned property.

Following are the few examples of these special Pythagorean Triangles, not all of them primitives, in Table 6a.1, when $0 < \beta < 25$. Table 6a.2 gives the values of X^2, Y^2 and Z^2; Table 6a.3 verifies $X^2 + Y^2 = Z^2$ and $X + Y = \beta^9$ while Table 6a.4 gives the verification of $X + Y + Z = \beta^9 + m^2 + n^2$.

Table 6a.1: Special Pythagorean Triangles

S.N.	M	N	X	Y	Z
1	4508	3969	20876702224609	1280530726526016	36075025
2	4802	2401	299096375126409	531726889113616	28824005
3	5488	1029	844443090845809	127561302844416	31176985
4	6104	259	1383225152310225	9997434544384	37325897
5	101808	93520	2620767481202999296	362604174327211622400	19110859264
6	104272	71344	33439430172846587904	221365689606683754496	15962616320
7	115248	38416	13938901361979975270 4	78406320161137360896	14757890560
8	126160	20464	24017468532742265241 6	26661489359513190400	16335120896
9	135632	86243	33568344677443593830 4	5472702227967901696	18470412800
10	244307	216308	16632640263458557822 5	1117061223638249922054 4	106475061113
11	250563	167042	12165297968916717120 25	700721163009602906126 4	90684846733
12	263167	122008	29561967051329027306 25	4123817781081356003584	84142821953
13	282931	79186	54434187762500282672 25	2007788080241847182224	86320373357
14	949026	936859	52649885197928821202 5	3162020485753266287374224	1778355134557

Table 6a.2: Values of X^2, Y^2 and Z^2

S.N.	X^2	Y^2	Z^2
1	20876702224609	1280530726526016	1301407428750625
2	299096375126409	531726889113616	830823264240025
3	844443090845809	127561302844416	972004393690225
4	1383225152310225	9997434544384	1393222586854609
5	2620767481202999296	362604174327211622400	365224941808414621696
6	33439430172846587904	221365689606683754496	254805119779530342400
7	139389013619799752704	78406320161137360896	217795333780937113600
8	240174685327422652416	26661489359513190400	266836174686935842816

9	335683446774435938304	5472702227967901696	341156149002403840000
10	166326402634585578225	11170612236382499220544	1133693863901708479876 9
11	121652979689167171202 5	7007211630096029061264	8223741426987700773289
12	295619670513290273062 5	4123817781081356003584	7080014486214258734209
13	544341877625002826722 5	2007788080241847182224	7451206856491875449449
14	526498851979288212025	3162020485753266287374 224	3162546984605245575586 249

Table 6a.3: Verification of $X^2 + Y^2 = Z^2$ and $X + Y = \beta^9$

S.N.	$X^2 + Y^2 = Z^2$	β	β^9	X + Y
1	1301407428750625	7	40353607	40353607
2	830823264240025	7	40353607	40353607
3	972004393690225	7	40353607	40353607
4	1393222586854609	7	40353607	40353607
5	365224941808414621696	14	20661046784	20661046784
6	254805119779530342400	14	20661046784	20661046784
7	217795333780937113600	14	20661046784	20661046784
8	266836174686935842816	14	20661046784	20661046784
9	341156149002403840000	14	20661046784	20661046784
10	11336938639017084798769	17	118587876497	118587876497
11	822374142698770077 3289	17	118587876497	118587876497
12	708001448621425873 4209	17	118587876497	118587876497
13	745120685649187544 9449	17	118587876497	118587876497
14	316254698460524557 5586249	23	1801152661463	1801152661463

Table6a.4: Verification of $X + Y + Z = \beta^9 + m^2 + n^2$

S.N.	β^9	m^2	n^2	$\beta^9 + m^2 + n^2$	X + Y + Z
1	7^9	4508^2	3969^2	76428632	76428632
2	7^9	4802^2	2401^2	69177612	69177612
3	7^9	5488^2	1029^2	71530592	71530592
4	7^9	6104^2	259^2	77679504	77679504
5	14^9	101808^2	93520^2	39771906048	39771906048
6	14^9	104272^2	71344^2	36623663104	36623663104
7	14^9	115248^2	38416^2	35418937344	35418937344
8	14^9	126160^2	20464^2	36996167680	36996167680
9	14^9	135632^2	8624^2	39131459584	39131459584
10	17^9	244307^2	216308^2	225062937610	225062937610
11	17^9	250563^2	167042^2	209272723230	209272723230
12	17^9	263167^2	122008^2	202730698450	202730698450
13	17^9	282931^2	79186^2	204908249854	204908249854
14	23^9	949026^2	936859^2	3579507796020	3579507796020

***One leg and Hypotenuse are Consecutive:**

We know that, in such cases, m = n + 1. Substituting the value of m in equations (6.2.2) and (6.2.4), we get the following nonic Diophantine equation,

$$2n^2 + 4n + 1 - \beta^9 = 0 \qquad \textbf{(6.2.7)}$$

Equation (6.2.7) was solved for $0 < n < 10^{13}$ and $0 < \beta < 10^{13}$ using the software Mathematica and it was found that it has no solution. It needs further investigations to find whether the solution of (6.2.7) exists for $n \geq 10^{13}$.

6.3 3D PLOTS

For $n < m$, $0 < n < 10^{12}$, $0 < m < 10^{12}$, $0 < \beta < 10^{7}$, we get 1100 Pythagorean triangles (X,Y,Z) with their perimeter as a sum of two squares and a nonic. Plotting these 1100 values for (X, Y, Z) as ListPlot3D (Figure 6a.1), ListSurfacePlot3D (Figure 6a.2) and ListPointplot3D (Figure 6a.3), we get following figures:

 ISBN: 978-93-93810-76-2

ListPlot3D

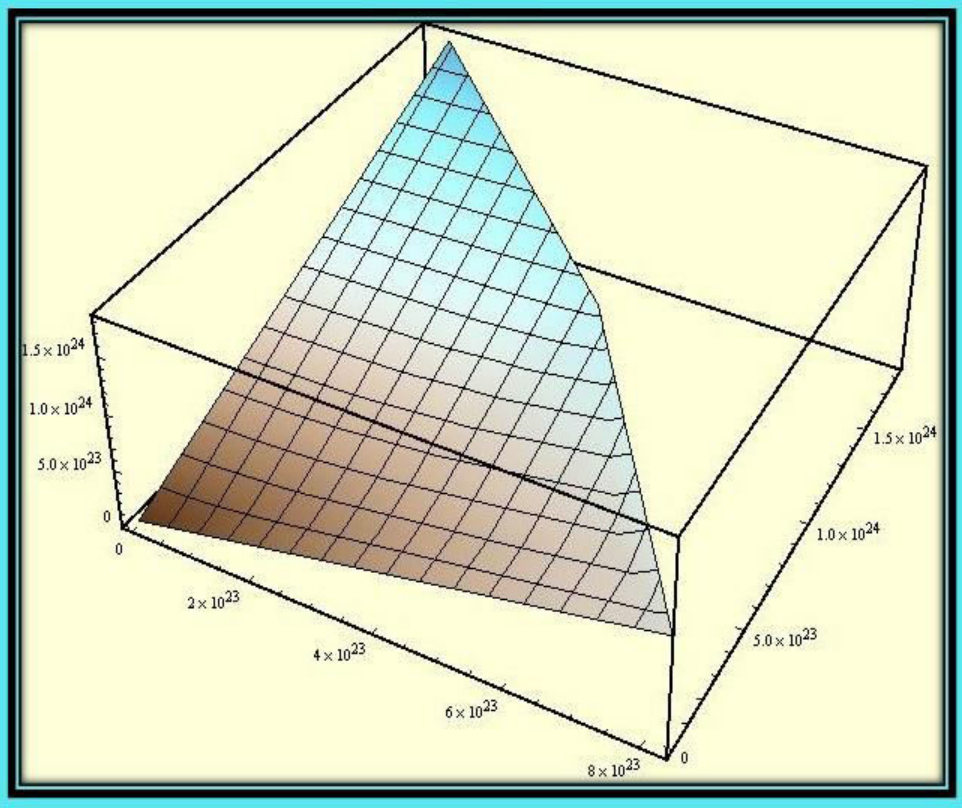

Figure 6a.1

ISBN: 978-93-93810-76-2

ListSurfacePlot3D

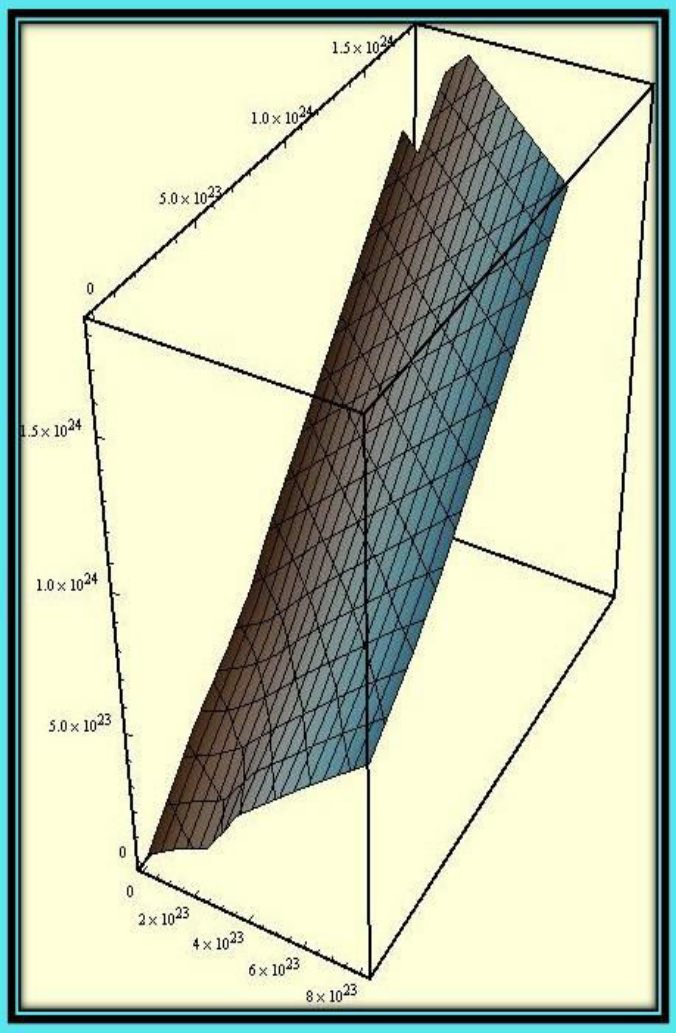

Figure 6a.2

ISBN: 978-93-93810-76-2

ListPointPlot3D

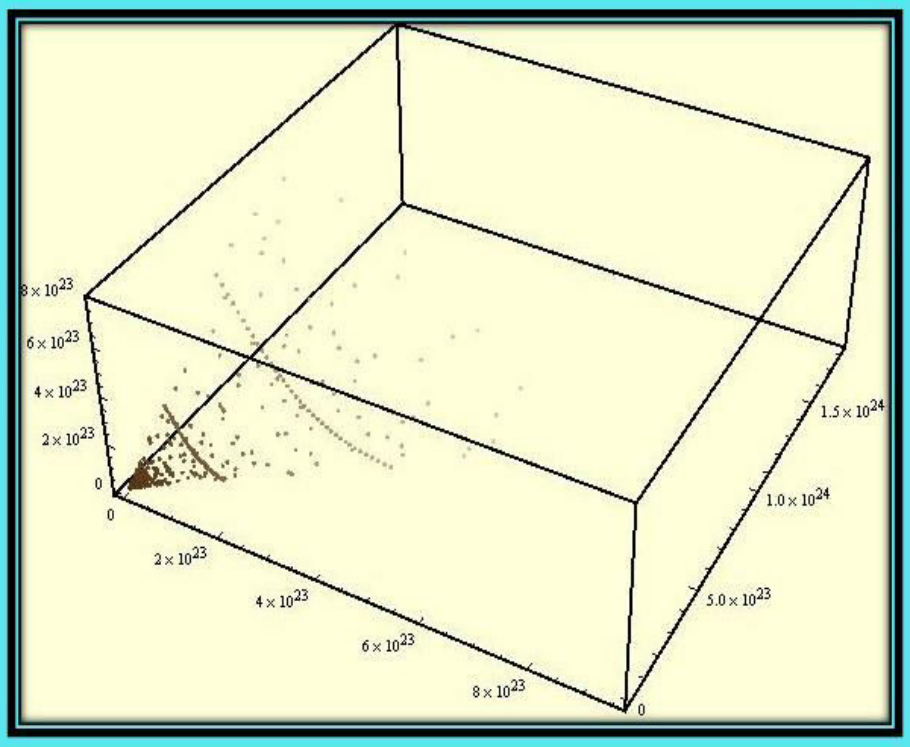

Figure 6a.3

6.4 OBSERVATIONS

For all Pythagorean Triangles (X, Y, Z), with

$X + Y + Z = m^2 + n^2 + \beta^9$, where $\beta < 100$, we made the following observations-

4. Except for $\beta = 49$ and $\beta = 98$, for each value of β, there are five pairs of (m, n) which will give five Pythagorean Triangles (X, Y, Z).
5. For $\beta = 49$ and $\beta = 98$, there are nine Pythagorean Triangles (X, Y, Z) correspondingly.
6. $X + Y + Z = 0 (\text{mod } 2)$.
7. $Y + Z - X = 0 (\text{mod } 2)$.

CHAPTER 6(B)

FINDING SPECIAL PYTHAGOREAN TRIANGLES WITH SUM OF TWO LEGS A DECIC THROUGH MATHEMATICA*

6.5 INTRODUCTION

In this chapter, we explore through decic Diophantine equation to discover special Pythagorean triangles. We examine whether there exists any Pythagorean Triangle with sum of its legs a decic. The existence of such triangles with two sides consecutive is also investigated. The patterns of Special Pythagorean Triangles with perimeter as sum of two squares and a decic are studied.

6.6 METHOD OF ANALYSIS

As we know that,

$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2,$ **(6.6.1)**

gives the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2,$ **(6.6.2)**

for some integers m, n of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

***Sum of two legs is a decic:** If the sum of legs X and Y is a decic, that is,

$X + Y = \beta^{10}.$ **(6.6.3)**

Then by values of X and Y from equation (6.6.1), we get,

$m^2 + 2mn - n^2 = \beta^{10}.$ **(6.6.4)**

Then, by theory of equations, the ten solutions of β, of equation (6.6.4), are given by,

$\beta = - (m^2 + 2mn - n^2)^{1/10},$

$\beta = (m^2 + 2mn - n^2)^{1/10},$

$\beta = - (-1)^{(1/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = (-1)^{(1/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = - (-1)^{(2/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = (-1)^{(2/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = - (-1)^{(3/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = (-1)^{(3/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = - (-1)^{(4/5)} (m^2 + 2mn - n^2)^{1/10},$

$\beta = (-1)^{(4/5)} (m^2 + 2mn - n^2)^{1/10}.$ **(6.6.5)**

Since we are seeking positive integral solutions for (X, Y, Z), from these values, only those of above values are taken which give β as a positive integer.

***Perimeter is sum of two squares and a decic:** If the sum of two legs is a decic and by equation (6.6.1), the hypotenuse is sum of two squares then, by equations (6.6.1) and (6.6.3), we get

* This paper is published in International Journal of Computational Mathematics and Numerical Simulation, 4(2)(2011);.279-284.

$X + Y + Z = \beta^{10} + m^2 + n^2$. **(6.6.6)**

Equations (6.6.1) and (6.6.4) generate X and Y which satisfy equations (6.6.2) and (6.6.3) in correspondence with equation (6.6.6).

Solving the Diophantine equation using software Mathematica, $m^2 + 2mn - n^2 - \beta^{10} = 0$, for $n < m$; $0 < n < 10^{13}$;

$0 < m < 10^{13}$; $0 < \beta < 10^9$; we get the 1468 values for (X, Y, Z) which will consume around 90 pages to write down! Out of these 1468 values only 52 are primitives. Following are the first ten examples (not necessarily primitives): Table 6b.1 gives the values of m, n, β , X, Y and Z; Table 6b.2 gives corresponding values of X^2 and Y^2; Table 6b.3 verifies $X^2 + Y^2 = Z^2$ and Table 6b.4 verifies $X + Y = \beta^{10}$ and $X + Y + Z = \beta^{10} + m^2 + n^2$.

Table 6b.1: First Ten Special Pythagorean Triangles (X, Y, Z)

S.N.	m	n	β	X	Y	Z
1	12005	9604	7	51883209	230592040	236356841
2	12467	7140	7	104446489	178028760	206405689
3	12985	5586	7	137406829	145068420	199813621
4	14063	3430	7	186003069	96472180	209532869
5	15025	2024	7	221654049	60821200	229847201
6	384160	307328	14	53128406016	236126248960	242029405184
7	398944	228480	14	106953204736	182301450240	211359425536
8	415520	178752	14	140704592896	148550062080	204609147904
9	450016	109760	14	190467142656	98787512320	214561657856
10	480800	64768	14	226973746176	62280908800	235363533824

Table 6b.2: Values of X^2 and Y^2

S.N.	X^2	Y^2
1	2691867376137681	53172688911361600
2	10909069064427121	31694239387137600
3	18880636655835241	21044846481296400
4	34597141677418761	9306881513952400
5	49130517438094401	3699218369440000
6	2822627525800944992256	5575560544791990108160 0
7	11438988003300732829696	3323381875960719605760 0
8	19797782462029093666816	22067120943971853926400
9	36277732431541054734336	9758972590374151782400
10	51517081453167274622976	3878911600953917440000

Table 6b.3: $X^2 + Y^2 = Z^2$

S.N.	Z^2	$X^2 + Y^2$
1	55864556287499281	55864556287499281
2	42603308451564721	42603308451564721
3	39925483137131641	39925483137131641
4	43904023191371161	43904023191371161
5	52829735807534401	52829735807534401
6	58578232973720846073856	58578232973720846073856
7	44672806762907928887296	44672806762907928887296
8	41864903406000947593216	41864903406000947593216
9	46036705021915206516736	46036705021915206516736
10	55395993054121192062976	55395993054121192062976

Table 6b.4: $X + Y = \beta^{10}$ & $X + Y + Z = \beta^{10} + m^2 + n^2$

S.N.	X + Y	β^{10}	X + Y + Z	$\beta^{10} + m^2 + n^2$
1	282475249	282475249	518832090	518832090
2	282475249	282475249	488880938	488880938
3	282475249	282475249	482288870	482288870
4	282475249	282475249	492008118	492008118
5	282475249	282475249	512322450	512322450
6	289254654976	289254654976	531284060160	531284060160
7	289254654976	289254654976	500614080512	500614080512
8	289254654976	289254654976	493863802880	493863802880
9	289254654976	289254654976	503816312832	503816312832
10	289254654976	289254654976	524618188800	524618188800

***One leg and hypotenuse are consecutive:** We know that, in such cases, m = n + 1.

Substituting the value of m = n + 1 (6.4.4), we get the decic Diophantine equation,

$$2n^2 + 4n+1- \beta^{10} = 0 \qquad \textbf{(6.6.7)}$$

Using the software Mathematica, equation (6.6.7) was solved for $0 < n < 10^{13}$ and $0 < \beta < 10^{13}$ and it was found that it has no solution. Whether the solution of decic equation (6.6.7) exists for $n \geq 10^{13}$, further investigations need to be carried out.

6.7 3D PLOTS

For $n < m$; $0 < n < 10^{13}$; $0 < m < 10^{13}$; $0 < \beta < 10^{9}$; we get the 1468 values for (X, Y, Z) out of which only 52 are primitives. Plotting these values as ListPlot3D (Figure 6b.1), ListSurfacePlot3D (Figure 6b.2) and ListPointPlot3D (Figure 6b.3), we get the following figures:

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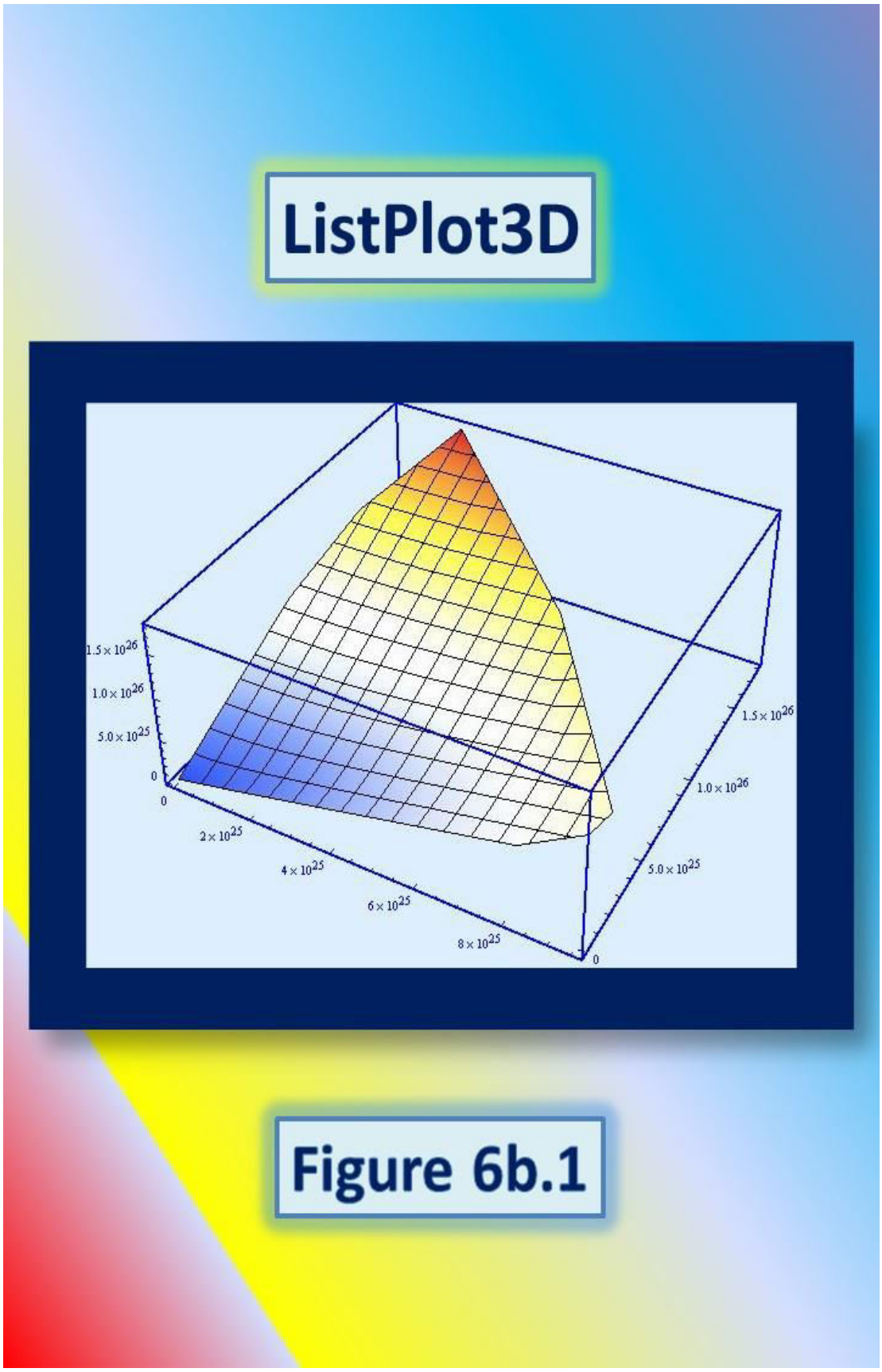

Figure 6b.1

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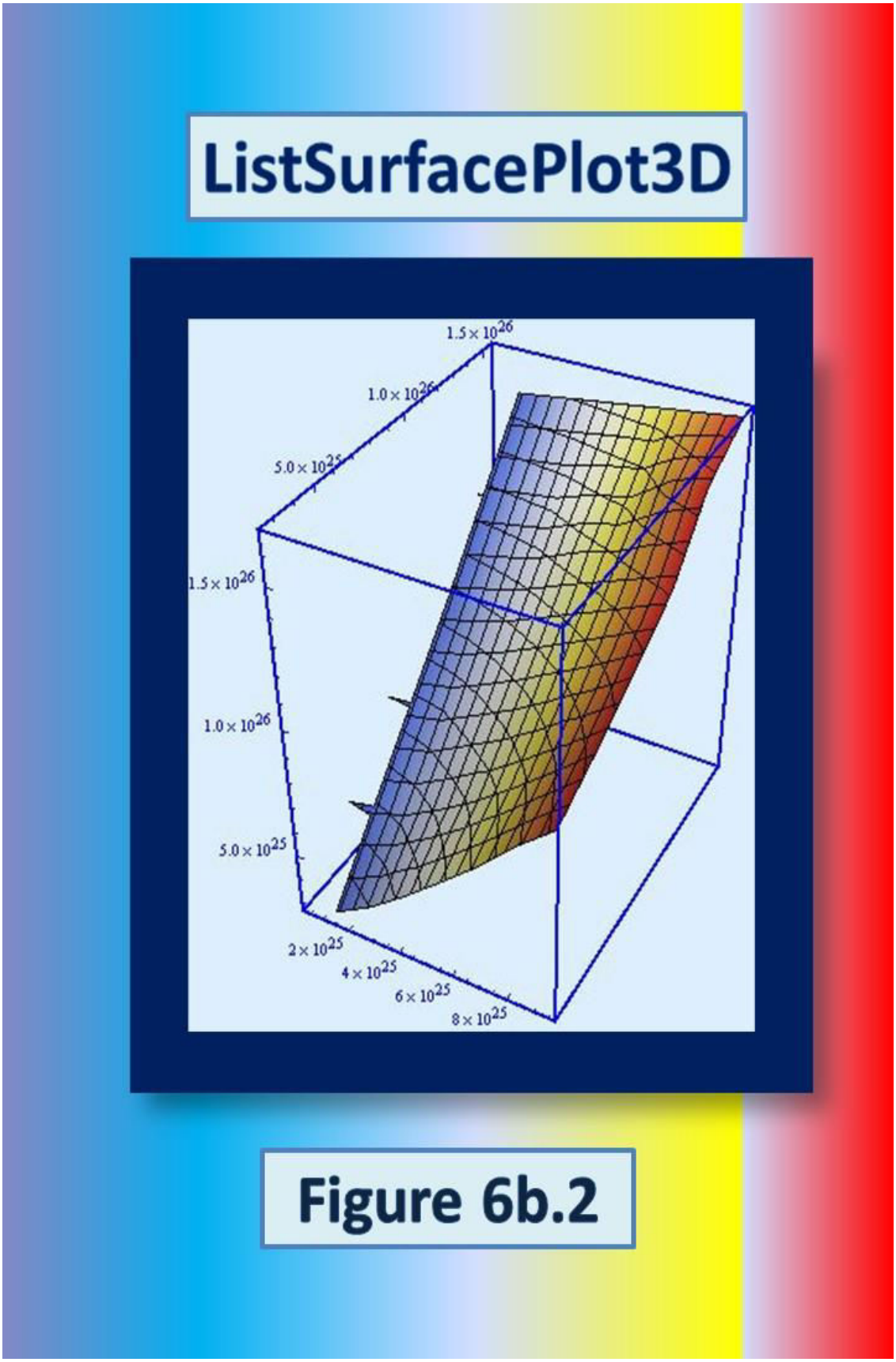

ISBN: 978-93-93810-76-2

ListPointPlot3D

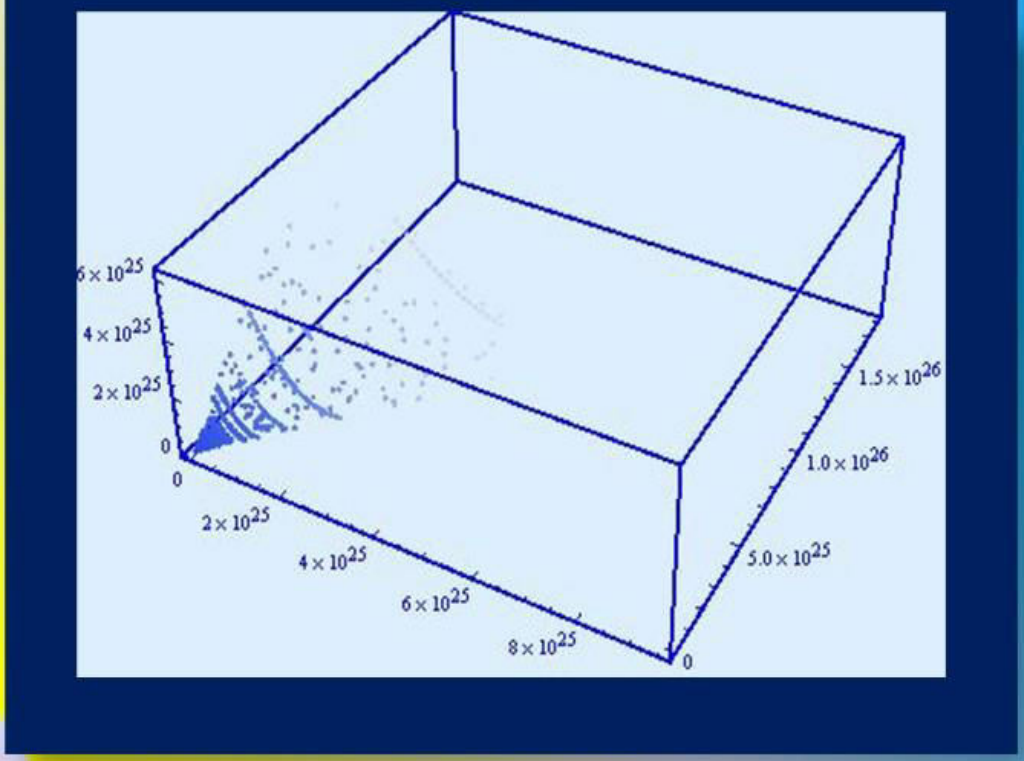

Figure 6b.3

6.8 OBSERVATIONS

With $\beta < 100$, all Pythagorean Triangles (X, Y, Z), which satisfy the equation (6.6.6), i.e., $X + Y + Z = m^2 + n^2 + \beta^9$,

We observe the following: -

1. Except for $\beta = 49$ and $\beta = 98$, for each value of β, there are five pairs of (m, n) which will give five Pythagorean Triangles (X, Y, Z).
2. For $\beta = 49$ and $\beta = 98$, there are ten Pythagorean Triangles (X, Y, Z) correspondingly.
3. $X + Y + Z = 0(\text{mod } 2)$.
4. $Y + Z - X = 0(\text{mod } 2)$.
5. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.
6. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.
7. $X + 2Y + Z \pm 2\{(X + Y)(Z - Y)\}^{1/2} = 0(\text{mod}16)$ or $= 0(\text{mod}4)$.

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ABOUT THE BOOK

Whenever Pythagoras is mentioned, we immediately think of his famous Pythagoras theorem of right-angled triangle that has applications in many branches of modern science: from geometry to Einstein's theory of General Relativity. The impact of this theorem was or rather still is so great that it outshines all the other contributions of this great teacher. In Pythagoras, who is considered as the father of pure Mathematics, one can see the union of mathematical genius and mysticism. Pythagorean triangles fascinate those who love numbers. More than 400 proofs of his great theorem have been discovered so far. In this book, Pythagoras and His Triangles, various aspects of Pythagoras teachings are touched upon and special Pythagorean Triangles are mentioned which satisfy eleven different constraints.

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